MA8353/TPDE UNIT - IV - FOURIER TRANSFORMS

1. Fourier integral theorem:

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds.$$

or equivalently

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt\, d\lambda.$$

This is known as Fourier integral theorem (or) Fourier integral formula.

2. For any non-zero real a, $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$

Proof:

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to -\infty$, if $a > 0$

$\frac{dt}{dx} = a$ $\quad x \to \infty \Rightarrow t \to \infty$, if $a > 0$

$dt = a\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a}\, \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{is/a\, x}\, dx \quad [\because t \text{ is dummy}]$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right] \to (1)$$

Similarly if $a<0$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to \infty$

$dt = a\,dx$, $x \to \infty \Rightarrow t \to -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{(is/a)t} \frac{dt}{a}$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{(is/a)t} \frac{dt}{a}$$

$$= \frac{-1}{a} F\left[\frac{s}{a}\right] \longrightarrow (2)$$

Combining (1) & (2), we get

$$F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right], \quad a \neq 0$$

3. Shifting property. (i) $F = [f(x-a)] = e^{ias} F(s)$

(ii) $F[e^{iax} f(x)] = F[s+a]$

proof :

i) WKT

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

put $t = x - a \Rightarrow x = t+a$, $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$, $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, e^{isa}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias} F[f(t)] = e^{isa} F[s]$$

(ii) WKT

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F[s+a].$$

4. Modulation theorem:

If $F(s)$ is the Fourier transform of $f(x)$,

then $F[f(x) \cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

Proof: WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

5. $F[x^n f(x)] = (-i)^n \dfrac{d^n F(s)}{ds^n}$

Proof:

WKT $F(s) = F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{isx} dx$

Differentiating both sides n times w.r.to s, we get

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n} [f(x) e^{isx}] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (ix)^n e^{isx} dx$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) x^n e^{isx} dx$$

$$= (i)^n F[x^n f(x)]$$

Hence $F[x^n f(x)] = \dfrac{1}{(i)^n} \dfrac{d^n}{ds^n} F(s)$

$$= (-i)^n \frac{d^n}{ds^n} F(s).$$

6. parseval's identity :.

I $F(s)$ is the Fourier transform of $f(x)$. Then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

Proof:

By convolution theorem

$F[f(x) * g(x)] = F(s) \cdot G(s)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx} ds \rightarrow (1)$$

put $x=0$, we get

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$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, G(s)\, ds \longrightarrow (2)$$

$g(-t) = \overline{f(t)}$ there it follows that

$$G(s) = \overline{F(s)}$$

$$(2) \Rightarrow \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

7. Find the (complex) Fourier transform of

$$f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a,\ x > b. \end{cases}$$

Soln:-

WKT $$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx}\, e^{isx}\, dx \longrightarrow (1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{i(k+s)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)x}}{i(k+s)}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{i(k+s)}\right]\left[e^{i(k+s)x}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i(k+s)}\left[e^{i(k+s)b} - e^{+i(k+s)a}\right]$$

$$= \frac{-i}{\sqrt{2\pi}\,(k+s)}\left[e^{i(k+s)b} - e^{i(k+s)a}\right]$$

$$= \frac{i}{\sqrt{2\pi}\,(k+s)}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$$

8. Convolution theorem for Fourier transforms.

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms

$$F[f(x) * g(x)] = F\{f(x)\} F\{g(x)\}.$$

9. $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$

Proof:

WKT $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

Differentiating both sides w.r to "s"

$$\frac{d}{ds} F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) (-x \sin sx) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, x \sin sx \, dx$$

$$= -F_s[x f(x)]$$

ie $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$

10. $F_c[x f(x)] = \frac{d}{ds} F_s(s).$

proof:

WKT $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

Diff both sides w. r to "s" we get

$$\frac{d}{ds} F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \sin sx \, dx$$

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$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\sin sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, (x) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, x \cos sx \, dx$$

$$= F_c [x f(x)]$$

ie. $F_c [x f(x)] = \frac{d}{ds} F_s [f(x)]$.

11. Find the Fourier Cosine transform of

proof

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

WKT $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[x \frac{\sin sx}{s} - (1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^1 + \left[(2-x) \frac{\sin sx}{s} - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[x \frac{\sin sx}{s} + \frac{\cos sx}{s^2} \right]_0^1 + \left[(2-x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{1 \sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right] + \left[\left(0 - \frac{\cos 2s}{s^2} \right) - \left(- \left(\frac{\sin s}{s} - \frac{\cos s}{s^2} \right) \right) \right] \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2} \right].$$

UNIT - IV - Fourier Transforms

1* Fourier Sine and Cosine integrals:-

i) $f(x) = \frac{2}{\pi} \int_0^\infty \sin ux \, du \int_0^\infty f(t) \sin ut \, dt$

[Fourier sine integral]

ii) $f(x) = 2/\pi \int_0^\infty \cos ux \, du \int_0^\infty f(t) \cos ut \, dt.$

[Fourier cosine integral]

2* Fourier Transform: [complex Fourier Transform]

The complex (or infinite) fourier Transform of $f(x)$ is given by $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$

3 * Fourier inverse Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \, e^{-isx} \, ds$

4* Parseval's identity for Fourier sine and Cosine transforms,

If $F_s[f(x)] = F_s[s]$ and $F_c[g(x)] = F_c(s)$ then

(i) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$ (sine)

(ii) $\int_0^\infty |g(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds.$ (cosine)

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Sine & Cosine Transforms:

5. Fourier Cosine Transform:.

The infinite Fourier cosine transform of $f(x)$ is defined by $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

* Inverse Fourier cosine transform $F_c[f(x)]$ is defined by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$.

6*. Fourier Sine Transform:

The infinite Fourier Sine transform of $f(x)$ is defined by $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

* The inverse Fourier sine transform of $F_s[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds.$$

7. Show that $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral

Soln: $\int_0^\infty |f(x)| \, dx = \int_0^\infty 1 \, dx = [x]_0^\infty = \infty$

and this value tends to ∞ as $x \to \infty$

i.e., $\int_0^\infty |f(x)| \, dx$ is not convergent

Hence $f(x) = 1$ cannot be represented by a Fourier integral.

8. Find Fourier cosine transform of e^{-x}

Soln:

WKT $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2} \right]$$

$\because$ formula

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

9. Find the Fourier sine transform of e^{-3x}

Soln:-

$$F_s[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x} \sin sx \, dx$$

$$= \frac{2}{\pi} \left[\frac{s}{s^2+3^2} \right]$$

$\because F_s[e^{-ax}] = \frac{2}{\pi} \int_0^\infty e^{-ax} \sin sx \, dx$

10. Find the Fourier sine tranform of $f(x) = e^{-x}$

Soln:

WKT $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2} \right]$$

$\because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$