Lesson 9

Taylor's expansion for function of two variables

9.1 Introduction

Let $z = f(x,y)$ which is continuous, together with all its partial derivatives up to $(n+1)$-th order inclusive, in some neighborhood of a point (a,b). Then like a function of single variable we can represent $f(x,y)$ as sum of an n-th degree polynomial in power of $(x-a)$ and $(y-b)$ and some remainder. We consider here in case $n=2$ and show that $f(x,y)$ has of the form

$$f(x,y) = A_0 + D(x-a) + E(y-b)$$

$$+\frac{1}{2!}[A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2] + R_2 \quad (1)$$

where A_0, D, E, A, B, C are independent of x and y, and R_2 is the remainder, and it is very similar to function of single variable.

Let us apply the Taylor formula for function $f(x,y)$ of the variable y assuming x to be constant.

$$f(x,y) = f(x,b) + \frac{y-b}{1} f_y(x,b)$$

$$+\frac{(y-b)^2}{1.2} f_{yy}(x,b) + \frac{(y-b)^3}{1.2.3} f_{yyy}(x,\eta_1) \quad (2)$$

where $\eta_1 = b + \theta_1(y-b)$, $0 < \theta_1 < 1$. We expand the functions $f(x,b)$, $f_y(x,b)$, $f_{yy}(x,b)$ in a Taylor's series in powers of $(x-a)$

$$f(x,b) = f(a,b) + \frac{x-a}{1} f_x(a,b)$$

$$+\frac{(x-a)^2}{1.2} f_{xx}(a,b) + \frac{(x-a)^3}{1.2.3} f_{xxx}(\xi_1,b) \quad (3)$$

where $\xi_1 = x + \theta_2(x-a)$, $0 < \theta_2 < 1$

$$f_y(x,b) = f_y(a,b) + \frac{x-a}{1} f_{yx}(a,b)$$

$$+\frac{(x-a)^2}{1.2} f_{yxx}(\xi_2,b) \quad (4)$$

where $\xi_2 = x + \theta_3(x-a)$, $0 < \theta_3 < 1$

$$f_{yy}(x,b) = f_{yy}(a,b) + \frac{x-a}{1} f_{yyx}(\xi_3,b) \quad (5)$$

where $\xi_3 = x + \theta_4(x-a)$, $0 < \theta_4 < 1$. Substituting expression (3), (4) and (5) into formula (2), we get

$$f(x,y) = f(a,b) + \frac{x-a}{1} f_x(a,b) + \frac{(x-a)^2}{1.2} f_{xx}(a,b)$$

$$+ \frac{(x-a)^3}{1.2.3} f_{xxx}(\xi_1,b) + \frac{y-b}{1}[f_y(a,b) + \frac{x-a}{1} f_{yx}(a,b)$$

$$+ \frac{(x-a)^2}{1.2} f_{yxx}(\xi_2,b)] + \frac{(y-b)^2}{1.2}[f_{yy}(a,b)$$

$$+ \frac{x-a}{1} f_{yyx}(\xi_3,b)] + \frac{(y-b)^3}{1.2.3} f_{yyy}(x,\eta_1),$$

arranging the numbers as given in (1), we have

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)$$

$$+(y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx} f(\xi_1,b)$$

$$+3(x-a)^2(y-b)f_{xxy}(\xi_2,b) + 3(x-a)(y-b)^2 f_{xyy}(\xi_3,b)$$

$$(y-b)^3 f_{yyy}(a,\eta_1)]$$

This is the Taylor's formula for $n = 2$. The expression

$$R_2 = \frac{1}{3!}[(x-a)^3 f_{xxx}(\xi_1, b) + 3(x-a)^2(y-b)f_{xxy}(\xi_2, b)$$

$$+3(x-a)(y-b)^2 f_{xyy}(\xi_3, b) + (y-b)^3 f_{yyy}(a, \eta_1)].$$

This is called the remainder. If we denote $x - a = \Delta x$, $y - b = \Delta y$, and $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, R_2 becomes

$$R_2 = \frac{1}{3!}[\frac{\Delta x^3}{\Delta\rho^3} f_{xxx}(\xi_1, b) + 3\frac{\Delta x^2 \Delta y}{\Delta\rho^3} f_{xxy}(\xi_2, b)$$

$$+3\frac{\Delta x \Delta y^2}{\Delta\rho^3} f_{xyy}(\xi_3, b) + \frac{\Delta y^3}{\Delta\rho^3} f_{yyy}(a, \eta_1)]\Delta\rho^3.$$

Example 9.1: Find the remainder R_2 of the function given by

$$f(x, y) = \sin x \sin y \ \ about \ \ (0,0)$$

Solution:

$$f(x, y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$

$$+\frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_y y(0,0)] + R_2.$$

Where R_2 is given by

$$R_2 = \frac{1}{3!}[(x-a)^3 f_{xxx} f(\xi_1, b) + 3(x-a)^2(y-b) f_{xxy}(\xi_2, b)$$

$$+3(x-a)(y-b)^2 f_{xyy}(\xi_3, b) + (y-b)^3 f_{yyy}(a, \eta_1)].$$

$$f_x(x,y) = \cos x \sin y, f_y(x,y) = \sin x \cos y,$$

$$f_{xx}(x,y) = -\sin x \sin y, f_{xy}(x,y) = \cos x \cos y,$$

$$f_{yy}(x,y) = -\sin x \sin y, f_{yx}(x,y) = \cos x \cos y,$$

$$f_{xxx} = -\cos x \sin y, f_{xxy} = -\sin x \cos y,$$

$$f_{xyy} = -\cos x \sin y, f_{yyy} = -\sin x \cos y$$

$$R_2 = \frac{1}{3!}[0 + 3x^2 y(-\sin(x + \theta_3 x))]$$

$$= -\frac{1}{2}[x^2 y \sin(x + \theta_3 x)]$$

Questions: Answer the following question.

1. Expand $z = \sin x \sin y$ in powers of $(x - \frac{\pi}{4})$ and $(y - \frac{\pi}{4})$. Find the terms of the first and second orders and R_2 (the remainder of second order).

2. Let $f(x,y) = e^x \sin y$. Expand $f(x+h, y+k)$ in powers of h and k and also find R_2.

3. Expand $x^2 y + \sin y + e^x$ in powers of $(x-1)$ and $(y-\pi)$ through quadratic terms and write the remainder.

4. Expand $x^3 - 2xy^2$ in Taylor's Theorem about $a=1$, $b=-1$.

5. Show that for $0<\theta<1$,

$$e^{ax}\sin by = by + abxy + \frac{1}{6}[(a^3x^3 - 3ab^2xy^2)\sin(b\theta y) + (3a^2bx^2y - b^3y^3)\cos(b\theta y)]e^{a\theta x}.$$

Keywords: Taylor's polynomial

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Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

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Suggested Readings

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