Lesson 8

Derivative of Higher Order

8.1 Introduction

Derivative of higher order of composite function may be computed by the principles given in Lesson 7. As an example, let us compute three drivatives of order two for the function $u = f\big(\varphi(r,s),\psi(r,s)\big)$. We assume that three

functions along with partial derivatives are continous upto order 3. First let us consider the higher order partial derivatives.

8.1.1 For $u = f(\emptyset(r,s),\psi(r,s)$, we assume that the three fucntions

$f,\emptyset,\psi \in C^2$.

$\frac{\partial u}{\partial r} = f_1\emptyset_1 + f_2\psi_1$, $\frac{\partial u}{\partial s} = f_1\emptyset_2 + f_2\psi_2$

Differentiating again, remember that f_1 *and* f_2 are themselves composite

functions.

$$\frac{\partial^2 u}{\partial r^2} = f_1\emptyset_{11} + f_2\psi_{11+}\emptyset_1[f_{11}\emptyset_1 + f_{12}\psi_1] + \psi_1[f_{21}\emptyset_1 + f_{22}\psi_1]$$

$$\frac{\partial^2 u}{\partial s\partial r} = f_1\emptyset_{12} + f_2\psi_{12+}\emptyset_1[f_{11}\emptyset_2 + f_{12}\psi_2] + \psi_1[f_{21}\emptyset_2 + f_{22}\psi_2]$$

$$\frac{\partial^2 u}{\partial s^2} = f_1 \emptyset_{22} + f_2 \psi_{22} + \emptyset_2 [f_{11}\emptyset_2 + f_{12}\psi_1] + \psi_2 [f_{21}\emptyset_1 + f_{22}\psi_2]$$

We omit the arguments in these fucntions to have space. If we admit that

$f_{12} = f_{21}, \emptyset_{12} = \emptyset_{21}, \psi_{12} = \psi_{21}$ then it is easily shown that $\frac{\partial^2 u}{\partial r \partial s} = \frac{\partial^2 u}{\partial s \partial r}$.

8.1.1 Higher-order partial derivatives As is true for ordinary derivatives, it is possible to take second, third, and higher order partial derivatives of a function of several variables, provided such derivatives exist.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}.$$

It is not true in general $f_{yx} = f_{xy}$

Example 8.1 Let $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$; for $(x, y) \neq (0,0)$ and $f(0,0) = 0$.

Solution:

We have

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(0+\Delta x,0) - f_y(0,0)}{\Delta x}$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, 0+\Delta y) - f(\Delta x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta x \Delta y (\Delta x^2 - \Delta y^2)}{\Delta y (\Delta x^2 + \Delta y^2)} = \Delta x.$$

Hence

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1.$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,0+\Delta y) - f_x(0,0)}{\Delta y}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x,0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$f_x(0,\Delta y) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, \Delta y) - f(0,\Delta y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \Delta y (\Delta x^2 - \Delta y^2)}{\Delta x (\Delta x^2 + \Delta y^2)} = -\Delta y$$

So

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} = -1.$$

i.e., $f_{yx}(0,0) \neq f_{xy}(0,0)$.

8.1.2 Partial Derivatives of Higher Order (Equality of f_{xy} and f_{yx}).

If $f(x,y)$ possesses continuous second order partial derivatives f_{xy} and f_{yx}, then

$$f_{xy} = f_{yx}$$

Note: Existence of partial derivatives does not ensure continuity of a function.

Example 8.2 Let $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$; for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

Solution:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

But $f(x,y)$ is discontinuous at $(0,0)$.

Example 8.3 If $f(x,y) = g(x)h(y)$, show that $f_{xy} = f_{yx}$

Solution:

$$f_x(x,y) = g'(x)h(y), f_{yx} = g'(x)h'(y)$$

$$f_y(x,y) = g(x)h'(y), f_{xy} = g'(x)h'(y)$$

i.e., $f_{xy} = f_{yx}$.

Example 8.4 If $z = g(x)h(y)$, show that $z\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$

Solution: We have $\frac{\partial z}{\partial x} = g'(x)h(y)$ and $\frac{\partial z}{\partial y} = g(x)h'(y)$. Now $\frac{\partial^2 z}{\partial x \partial y} = g'(x)h'(y)$.

So

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = g(x)h(y)g'(x)h'(y)$$

$$= z\frac{\partial^2 z}{\partial x \partial y}$$

Example 8.5 Let $f(x,t) = u(x+at) + v(x-at)$, where u and v are assumed to have continuous second partial derivatives, show that $a^2 f_{xx} = f_{tt}$.

Solution:

$$f_x = u'(x+at) + v'(x-at), f_{xx} = u''(x+at) + v''(x-at)$$

$$f_t = au'(x+at) - av'(x-at), f_{tt} = a^2u''(x+at)$$

$$+a^2v''(x-at) = a^2 f_{xx}.$$

Questions: Answer the following questions.

1. For $u = f(g(t), h(t))$, find $\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial t^2}$

2. Find $f''(t)$, if $f = e^x \sin y, x = t^2, y = 1 - t^2$ by not eleminating x and y.

3. Show that the functions $z = \phi(x^2 - y^2)$, where $\phi(u)$ is a differentiable function, satisfies the relationship $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$.

4. Find the derivatives $\frac{dy}{dx}$ of the functions represented implicitly

   $(i) \sin(xy) - e^{xy} - x^2 y = 0$ $(ii)\ xe^y + ye^x - e^{xy} = 0$ $(iii)\ y^x = x^y$ $(iv)\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

5. If $r = x\phi(x+y) + y\psi(x+y)$, show that

$$\frac{\partial^2 r}{\partial x^2} - 2\frac{\partial^2 r}{\partial x \partial y} + \frac{\partial^2 r}{\partial y^2} = 0.$$

(ϕ and ψ are twice differentiable function.)

6. If $u = \frac{1}{y}[\phi(ax+y) + \phi(ax-y)]$, show that

$$\frac{\partial^2 u}{\partial x^2} = \frac{a^2}{y^2} \cdot \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right).$$

Keywords: Higher order derivatives, higher order partial derivatives

References

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Suggested Readings

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