Lesson 7

Composite and Implicit Functions for Two Variables

7.1 Introduction

The chain rule works for functions of more than one variable. Consider the function $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, and $g(t)$ and $h(t)$ are differentiable with respect to t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that each argument of $z = F(u, v)$ is a two-variable function such that $u = h(x, y)$ and $v = g(x, y)$, and that these functions are all differentiable. Then the chain rule would look like:

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

If we consider $\vec{r} = (u, v)$ above as a vector function, we can use vector notation to write the above equivalently as the dot product of the gradient of F and a derivative of $\vec{r}$:

$$\frac{\partial F}{\partial x} = \nabla F . \partial\vec{r}$$

Partial and Total Increment: We consider a function $z = f(x, y)$, increase the independent variable x by Δx (keeping y fixed), then z will be increased: this increase is called the partial increment with respect to x which we denote as $\Delta_x z$, so that

$$\Delta_x z = f(x + \Delta x, y) - f(x, y).$$

Similarly we define $\Delta_y z$. If we increase the argument x by Δx and y by Δy, we get z a new increment Δz, which is called the total increment of z and defined by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

It is noted that total increment is not equal to the sum of the partial increments, $\Delta z \neq \Delta_x z + \Delta_y z$. Let us assume that $f(x, y)$ has continuous partial derivatives at the point (x, y) under consideration. Express Δz in terms of partial derivatives. To do this we have

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)]$$

$$+[f(x, y + \Delta y) - f(x, y)]$$

and using Lagrange mean value theorem separately

$$\Delta z = \Delta x \frac{\partial f(\bar{x}, y + \Delta y)}{\partial x} + \Delta y \frac{\partial f(x, \bar{y})}{\partial y}.$$

(where $\bar{y}$ lies between y and $y + \Delta y$ and $\bar{x}$ between x and $x + \Delta x$). As partial derivatives are continuous it follows that

$$\Delta z = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y + \psi_1 \Delta x + \psi_2 \Delta y.$$

Where the quantities $\psi_1(x,y)$ and $\psi_2(x,y)$ approach zero as Δx and Δy approach zero.

Now we will derive the total differential of composite function.

Theorem 7.1: $f(x,y), g(r,s), h(r,s) \in C^1$

$$\Rightarrow \frac{\partial}{\partial r} f(g,h) = f_1(g,h) g_1(r,s) + f_2(g,h) h_1(r,s)$$

$$\frac{\partial}{\partial s} f(g,h) = f_1(g,h) g_2(r,s) + f_2(g,h) h_2(r,s).$$

We use this formula for the composite function $f(x,y), x = \phi(r,s), y = \psi(r,s)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Example 7.1: $f(x,y) = xy, f_1 = y, f_2 = x$

$$\frac{\partial}{\partial r} gh = \frac{\partial}{\partial r} f(g(r,s), h(r,s))$$

$$= y g_1 + x h_1$$

$$= h g_1 + g h_1$$

We can generalize this results. If $w = F(z, u, v, s)$ is a function of four arguments z, u, v, s and each of them depends on x and y, then

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial y}.$$

If a function $z = F(x, y, u, v)$, where y, u, v depend on a single independent variables x: $y = f(x)$, $u = \phi(x)$, $v = \psi(x)$, then z is actually a function of one variable x only.

Hence,

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}$$

$$+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$+\frac{\partial F}{\partial u}\frac{du}{dx} + \frac{\partial F}{\partial v}\frac{dv}{dx}$$

This formula is known as the formula for calculating the total derivative $\frac{dz}{dx}$ (in contrast to the partial derivative $\frac{\partial z}{\partial x}$).

Example 7.2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $z = \ln(u^2 + v)$, $u = e^{x+y^2}$, $v = x^2 + y$.

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial x} = e^{x+y^2}, \frac{\partial u}{\partial y} = 2ye^{x+y^2}, \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 1, \frac{\partial z}{\partial u} = \frac{2u}{u^2+v}, \frac{\partial z}{\partial v} = \frac{1}{u^2+v}.$$

So

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x$$

$$= \frac{2}{u^2+v}(ue^{x+y^2} + x)$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v}$$

$$= \frac{1}{u^2+v}(4uye^{x+y^2} + 1).$$

In these expressions, we have to substitute e^{x+y^2} and $x^2 + y$ for u and v respectively.

Example 7.3: Find the total derivative of $z = x^2 + \sqrt{y}, y = \sin x$

Solution:

$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}}, \frac{dy}{dx} = \cos x.$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= 2x + \frac{1}{2\sqrt{y}}\cos x$$

$$= 2x + \frac{1}{2\sqrt{\sin x}}\cos x.$$

7.1.1 Let us find the the total differential of the composite function $z = F(u, v)$ and $u = \phi(x, y)$ and $v = \psi(x, y)$, we know the total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Now substitute the expression $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ defined in the above composite function, after simplification we obtain

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

Where $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ and $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

Example 7.4: Find the total differential of the composite function $z = u^2v^3$, $u = x^2\sin y$, $v = x^3e^y$.

Solution:

$$du = 2x\sin y dx + x^2\cos y dy$$

$$dv = 3x^2e^y dx + x^3e^y dy$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

$$= 2uv^3du + 3u^2v^2dv$$

$$= 2uv^3(2x\sin y dx + x^2\cos y dy)$$

$$+3u^2v^2(3x^2e^ydx + x^3e^y)dy$$

$$= (2uv^3.2x\sin y + 3u^2v^2.3x^2e^y)dx$$

$$+(2uv^3x^2\cos y + 3u^2v^2x^3e^y)dy$$

7.2 Composite and implicitly Functions:

Let some function y of x be defined by the equation $F(x, y) = 0$. We shall prove the following theorem.

Theorem 7.2 Let a function y of x be defined implicitly by the equation

$$F(x, y) = 0 \qquad (7.1)$$

where $F(x, y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are continuous in the domain D containing the point (x, y), which satisfies (7.1), also $\frac{\partial F}{\partial y} \neq 0$ at the point (x, y). Then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Proof. Given $F(x, y)$ is a function of two variables x, and y and y is again a function of x so that F is a composite function of x. Its derivative with respect to x is

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

As F is considered as a function of x alone, which is identically zero. So we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

which implies $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$.

Example 7.5: An equation is given that connects x and y

$$e^y - e^x + xy = 0,$$

find $\frac{dy}{dx}$.

Solution:

$F(x,y) = e^y - e^x + xy$, $\frac{\partial F}{\partial x} = -e^x + y$, $\frac{\partial F}{\partial y} = e^y + x$, by the above theorem we obtain $\frac{dy}{dx} = -\frac{-e^x + y}{e^y + x}$.

Questions: Answer the following questions.

1. Find $\frac{df}{dt}$ at $t = 0$ where $f(x,y) = x^3 + y^3$, $x = e^t$, $y = \cos t$.
2. If $z = f(x,y)$, $x = e^{2u} + e^{-2v}$,$y = e^{-2u} + e^{2v}$, then show that
3. $\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = 2[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}]$.
4. Find $\frac{dy}{dx}$, when $f(x,y) = \ln(x^2 + y^2) + tan^{-1}\left(\frac{y}{x}\right) = 0$.

5. Find $\frac{dy}{dx}$, when $x^y + y^x = a$, a any constant, $x, y > 0$.

Keywords: Chain Rule, Composite Function

References

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Suggested Readings

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