Lesson 6

Homogeneous Functions, Euler's Theorem

6.1 Introduction

A polynomial in x and y is said to be homogeneous if all its terms are of same degree. For example,

$$f(x,y) = x^2 - 2xy + 3y^2$$

is homogeneous. It is easy to generalize the property so that functions not polynomials can have this property.

Definition 6.1

A function $f(x,y)$ is homogeneous of degree n in a region D iff, for $(x,y) \in D$ and for every positive value λ, $f(\lambda x, \lambda y) = \lambda^n f(x,y)$. The number n is +ve, -ve, or zero and need not be an integer.

Example 6.1 $f(x,y) = x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}(\frac{y}{x})$. Here $n = -1$; D is any quadrant without the axes.

Example 6.2 $f(x,y) = 3 + \ln(\frac{y}{x})$

This function is homogeneous of degree 0; D is first and third quadrant without the axes.

Example 6.3 $f(x,y) = x^{\frac{1}{3}}y^{-\frac{2}{3}} + x^{\frac{2}{3}}y^{-\frac{1}{3}}$.

This function is not homogeneous.

Theorem 6.1 [Euler's Theorem] Let $f(x,y)$ is a homogeneous function of degree n in R (region) and f_x and f_y are continuous in R. Then

$$f_x(x,y)x + f_y(x,y)y = nf(x,y)$$

for all $(x,y) \in R$.

Proof. Now differentiate $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ partially with respect to λ, we obtain

Chain rule : $$xf_1(\lambda x, \lambda y) + yf_2(\lambda x, \lambda y) = n\lambda^{n-1}f(x,y).$$

Finally set $\lambda = 1$.

Example 6.4 If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$. Then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\cos 2u \sin u}{4\cos^3 u}$$

Proof. Let $w = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x,y)$.

u is not homogeneous function , but w is

$$f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})} = \frac{\lambda(x+y)}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})} = \lambda^{\frac{1}{2}} \frac{x+y}{\sqrt{x}+\sqrt{y}} = \lambda^{\frac{1}{2}} f(x,y)$$

w is homogeneous function of degree $\frac{1}{2}$.Therefore

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{1}{2}w = \frac{1}{2}\sin u$$

But $\frac{\partial w}{\partial x} = \cos u\frac{\partial u}{\partial x}, \frac{\partial w}{\partial y} = \cos u\frac{\partial u}{\partial y}$

Hence $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \qquad \text{............................(6.1)}$$

Differentiating (1) partially w.r.t. x , we have

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (\frac{1}{2}\sec^2 u - 1)\frac{\partial u}{\partial x} \qquad \text{..................(6.2)}$$

Differentiating (1) partially w.r.t. y , we have

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(\frac{1}{2}\sec^2 u-1)\frac{\partial u}{\partial y} \qquad \text{..................(6.3)}$$

Multiplying (2) by x , (3) by y and adding, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=(\frac{1}{2}\sec^2 u-1)(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y})$$

$$=(\frac{1}{2\cos^2 u}-1)(\frac{1}{2}\tan u)$$

$$=-\frac{\cos 2u\sin u}{4\cos^3 u}$$

Example 6.5 If $u=\sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$. Then find

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$$

Ans.: $(-7\tan u)$.

Example 6.6 (1) If $u = tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution:

Here u is not a homogenous function but $\tan u = \frac{x^3+y^3}{x-y}$ is a homogenous fucntion of degree 2

i.e., $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \,.\cos^2 u = \sin 2u$

(2) If $u = ln\frac{x^4+y^4}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Solution:

u is not homogenous function, but e^u is a homogenous function of degree 3 in x,y .

By Euler's theorem, we have $x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\text{i.e., } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

Questions: Answer the following questions.

1. If $u = sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$

2. If $= f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

3. If $= xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

4. If $u = sin^{-1}\frac{x}{y} + tan^{-1}\frac{y}{x}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Keywords: Homogeneous Function, Euler’s Theorem, Parial Derivatives

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Suggested Readings

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