# Lesson 5

# Partial and Total Derivatives

## 5.1 Introduction

Let $z = f(x, y)$, we denote $\frac{\partial z}{\partial x}$ as the partial derivative of $z$ with respect to $x$ and

define as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and similarly $$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

**Example 5.1:** Given $z = x^y$, find the partial derivative of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

**Solution:**

$\frac{\partial z}{\partial x} = yx^{y-1}$, $\frac{\partial z}{\partial y} = x^y \ln x$.

The partial derivatives of a function of any number of variables are determined similarly. Thus if $u = f(x, y, z, t)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z, t) - f(x, y, z, t)}{\Delta x}$$

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to 0} \frac{f(x,y,z,t+\Delta t) - f(x,y,z,t)}{\Delta t}$$

Informally, we say that the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(x_0, y_0, z_0)$ denote the slope of the surface in the $x$- and $y$-directions, respectively.

**Example 5.2:** Find the slopes of the surface given by $f(x,y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$ at the point $(\frac{1}{2}, 1, 2)$ in the $x$-direction and the $y$-direction.

**Solution:**

$$\frac{\partial f}{\partial x}\Big|_{(\frac{1}{2},1,2)} = -x\Big|_{(\frac{1}{2},1,2)} = -\frac{1}{2}$$

$$\frac{\partial f}{\partial y}\Big|_{(\frac{1}{2},1,2)} = -2y\Big|_{(\frac{1}{2},1,2)} = -2.$$

### 5.1.2 Differentiability for Functions of Two Variables

We begin by reviewing the concept of differentiation for functions of one variable. We define the derivative in case of function of single variable.

Let $f: D \subset R \mapsto R$ and let $a$ be an interior point of $D$. Then $f$ is differentiable at $a$ means

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

or equivalently

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

exists. The number $f'(a)$ is called the derivative of $f$ at $a$.

Geometrically the derivative of a function at $a$ is interpreted as the slope of the tangent line to the graph of $f$ at the point $(a, f(a))$.

Extending the definition of differentiability in its present form to functions of two variables is not possible because the definition involves division and dividing by a vector or by a point in two dimensional space is not possible. To carry out the extension, an equivalent definition is developed that involves division by a distance. The limit statement can be rewritten as

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}-f'(a)=0 \quad or$$

$$\lim_{x\to a}\frac{f(x)-f(a)-(x-a)f'(a)}{|x-a|}=0$$

So the following definition is equivalent to the original one.

Let $f: D \subset \mathrm{R} \mapsto R$ and let $a$ be an interior point of $D$. Then $f$ is differentiable at $a$ means there is a number, $f'(a)$, such that

$$\lim_{x \to a} \frac{f(x) - f(a) - (x-a)f'(a)}{|x-a|} = 0.$$

One way to interpret this expression is that $f(x) - f(a) - (x-a)f'(a)$ tends to 0 faster than $|x-a|$ and consequently $f(x)$ is approximately equal to $f(a) + (x-a)f'(a)$. The equation $y = f(a) + (x-a)f'(a)$ is the equation of the line tangent to the graph of $f$ at the point $(a, f(a))$. So $f(x)$ is approximated very well by its tangent line. This observation is the bases for linear approximation.

Using this form of the definition as a model it is possible to construct a definition of differentiability for functions of two variables.

**Definition 5.1**. Let $f: D \subset R^2 \mapsto R$ and let $(x_0, y_0)$ be an interior point of $D$.

Then f is differentiable at $(x_0, y_0)$ means there are two numbers,

$f_x(x_0, y_0) = f_x()$ and $f_y(x_0, y_0) = f_y()$ such that

$$\lim_{(x,y)\to(x_0,y_0)} = \frac{f(x,y) - f(x_0,y_0) - (x-x_0)f_x() - (y-y_0)f_y()}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

$$= 0.$$

The vector

$$f_x(x_0, y_0)\vec{\imath} + f_y(x_0, y_0)\vec{\jmath}\,,\ \vec{\imath} = (1,0),\ \vec{\jmath} = (0,1)$$

or

$$(f_x(x_0, y_0), f_y(x_0, y_0))$$

is called the derivative of $f$ at the point$(x_0, y_0)$. Interpret this definition as requiring that the graph of $f$ has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

In fact it is easy to get an equation for this tangent plane. It is $z = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$. In $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$, the same symbol $x$ is use for two different purposes. First as a subscript where it denotes the variable of differentiation and second as the first coordinate of a point in $\mathbb{R}^2$. Strictly speaking such a dual use of one symbol is improper, but this is so common as to be acceptable. In the general case, the derivative is a vector in $n$ space and it is computed by computing all of the first order partial derivatives. As in the case of functions of one variable, differentiability implies continuity.

For functions of one variable if the derivative, $f(x)$, can be computed, then $f$ is differentiable at $x$. The corresponding assertion for functions of two variables is false, as we know existence of partial derivative does not mean the function of two variable is continuous. We might suspect that if $f$ is continuous at $(x_0, y_0)$ and the first order partial derivatives exist there, then $f$ is differentiable at $(x_0, y_0)$ but that conjecture is false as the following example shows.

**Example 5.1.** Let $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$ if $(x, y) \neq (0,0)$ and $\mathrm{f}(0,0) = 0$.

**Solution:**

So if $f$ were differentiable at (0, 0),we would have that $lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$ as $f_x(0,0) = 0$ and $f_y(0,0) = 0$. That is

$lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$. But if the limit is computed along the path $y = x$, we get $lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$.

The natural question to ask then is under what conditions can we conclude that $f$ is differentiable at $(x, y)$. The answer is contained in the following theorem.

**Theorem 5.1.** Let $f: D \subset \mathbb{R}^2 \mapsto \mathbb{R}$ and let $P_0$ be an interior point of $D$. Suppose all of the first order partial derivatives of $f$ exist in a open disk about $P_0 = (x_0, y_0)$ and are continuous at $P_0$. Then $f$ is differentiable at $P_0$.

**Example 5.2.** Show that the function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

**Solution:**

The domain of $f$ is all of $\mathbb{R}^2$ except for the origin. We shall show that $f$ has continuous partial derivatives everywhere in its domain (that is, the function $f$ is in $C^1$). The partial derivatives are $f_x = \frac{2x}{x^2+y^2}$ and $f_y = \frac{2y}{x^2+y^2}$. Since each of $f_x$ and $f_y$ is the quotient of continuous functions, the partial derivatives are continuous everywhere except the origin (where the denominators are zero). Thus, $f$ is differentiable everywhere in its domain.

We know that if a function is differentiable at a point, it has partial derivatives there. Therefore, if any of the partial derivatives fail to exist, then the function cannot be differentiable. This is what happens in the following example.

**Example 5.3:** Consider the function $f(x,y) = \sqrt{x^2 + y^2}$. Is it differentiable at the origin.

**Solution:**

Let us find the partial derivatives if they exist at $(0,0)$. Now

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{\Delta^2 x + 0} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}.$$

Since the limit does not exit so $f_x(0,0)$ does not exit. Similarly we can show also $f_y(0,0)$ does not exist. Thus $f$ cannot be differentiable at the origin.

**In Example 5.3** the partial derivatives $f_x$ and $f_y$ did not exist at the origin and this was sufficient to establish non differentiability there.

In the following example even if both of the partial derivatives, $f_x(0,0)$ and $f_y(0,0)$, exist $f$ is not differentiable at (0, 0).

**Example 5.4:** Consider the function $f(x,y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but that $f$ is not differentiable at (0, 0).

**Solution:**

Now
$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0,$$

and similarly $f_y(0,0) = 0$. Suppose the function is differentiable at (0,0),

i.e.,
$$lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$$

That is
$$lim_{(x,y)\to(0,0)} \frac{x^{\frac{1}{3}}y^{\frac{1}{3}}}{\sqrt{x^2+y^2}} = 0.$$

If this limit exists, we get the same value no matter how $x$ and $y$ approach 0.

Suppose we take $y = x > 0$. Then the limit becomes

$$lim_{(x,y)\to(0,0)} \frac{x^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{x^2+y^2}}$$

$$= \lim_{x\to 0} \frac{x^{\frac{2}{3}}}{x\sqrt{2}} = \lim_{x\to 0} \frac{1}{x^{\frac{1}{3}}\sqrt{2}}.$$

But this limit does not exist, since small values for $x$ will make the fraction

arbitrarily large. Thus, this function is not differentiable at the origin, even though the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist.

In summary if a function is differentiable at point, then it is continuous there. Having both partial derivatives at a point does not guarantee that a function is continuous there.

**Theorem 5.1** : $f(x,y), g(r,s), h(r,s) \in C^1$

$$\Rightarrow \frac{\partial}{\partial r} f(g,h) = f_1(g,h)g_1(r,s) + f_2(g,h)h_1(r,s)$$

$$\frac{\partial}{\partial s} f(g,h) = f_1(g,h)g_2(r,s) + f_2(g,h)h_2(r,s).$$

Here subscript 1 and 2 denote the partial derivative with respect to its first and second argument, respectively. The proof is given in Lesson 7.

### 5.1.2 Total Differential

**Definition 5.2 (Total Differential)** For a function of two variables, $z = f(x, y)$

if $\Delta x$ and $\Delta y$ are given increments and, then the corresponding increment of $z$ is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The differentials $dx$ and $dy$ are independent variables; that is, they can be given any values. Then the differential $dz$, also called the total differential, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

**Example 5.5:** If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential $dz$.

Further, if $x$ changes from 2 to 2.05 and $y$ changes from 3 to 2.96, compare the values of $\Delta z$ and $dz$. Which is easier to compute $\Delta z$ or $dz$?

**Solution:**

By definition,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy.$$

Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of $z$ is $\Delta z = f(2.05,2.96) - f(2,3)$

$$[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(6) - 3^2]$$

$$= 0.6449$$

Notice that $\Delta z \approx dz$ but $dz$ is easier to compute.

**5.2 Total derivative:** In the mathematical field of differential calculus, the term total derivative has a number of closely related meanings.

The total derivative of a function, $f$, of several variables, e.g., $t$, $x$, $y$, etc., with respect to one of its input variables, e.g., $t$, is different from the partial derivative. Calculation of the total derivative of $f$ with respect to $t$ does not assume that the other arguments are constant while $t$ varies; instead, it allows the other arguments to depend on $t$. The total derivative adds in these indirect dependencies to find the overall dependency of $f$ on $t$. For example, the total derivative of $f(t, x, y)$ with respect to $t$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Consider multiplying both sides of the equation by the differential $dt$ .

The result will be the differential change $df$ in the function $f$. Because $f$ depends on $t$, some of that change will be due to the partial derivative of $f$ with respect to $t$. However, some of that change will also be due to the partial derivatives of $f$ with respect to the variables $x$ and $y$. So, the differential is applied to the total derivatives of $x$ and $y$ to find differentials $dx$ and $dy$, which can then be used to find the contribution to $df$.

**Example 5. 6:** Find the total derivative of $z = x^2 + \sqrt{y}, y = \sin x$

**Solution:**

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}}, \frac{dy}{dx} = \cos x.$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= 2x + \frac{1}{2\sqrt{y}}\cos x$$

$$= 2x + \frac{1}{2\sqrt{\sin x}}\cos x.$$

**Questions: Answer the following questions.**

1. Test the differentiability of $f(x, y) = \sqrt{y^2 - x^2}$

2. Find the total differential of $z = \tan^{-1}\left(\frac{x}{y}\right), (x, y) \neq (0,0)$

3. $u = xz + \frac{x}{z}$ , $z \neq 0$

4. Find $\frac{df}{dt}$ at $t = 0$ where $f(x,y) = x\cos y + e^x \sin y, x = t^2 + 1, y = t^3 + t$

**Keywords:** Partial Derivative, Differential, Total Differential

**References**

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## Suggested Readings

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