Lesson 40

Series Solutions about a Regular Singular Point (Cont...)

In this lesson we continue series solution about a singular point. We shall demonstrate the method with some useful differential equations.

40.1 Example Problems

40.1.1 Problem 1

Find one series solution of the differential equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0,$$

Solution: Note that $x = 0$ is a singular point. Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$y' = \sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1},$$
$$y'' = \sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2}.$$

Plugging into our equation we obtain

$$4\sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r} - 4\sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r+1} + (1-2x)\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Splitting the last series into two series we get

$$\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} - \sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1} + \sum_{k=0}^{\infty} a_k x^{k+r} - 2\sum_{k=0}^{\infty} a_k x^{k+r+1} = 0$$

Re-indexing leads to

$$\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\,a_k x^{k+r} - \sum_{k=1}^{\infty} 4(k+r-1)\,a_{k-1} x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} - \sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} = 0$$

Combining different series into one series

$$\big(4r(r-1)+1\big)\,a_0 + \sum_{k=1}^{\infty}\Big(\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1}\Big)\,x^{k+r}.$$

The indicial equation is given by

$$4r(r-1)+1=0$$

It has a double root at $r=\frac{1}{2}$. All other coefficients of x^{k+r} also have to be zero so

$$\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} = 0.$$

If we plug in $r=\frac{1}{2}$ and solve for a_k, we get

$$a_k = \frac{4(k+\frac{1}{2}-1)+2}{4(k+\frac{1}{2})\,(k+\frac{1}{2}-1)+1}\,a_{k-1} = \frac{1}{k}\,a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1}a_0 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2},$$
$$a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}, \ldots$$

In general, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = \sqrt{x}\sum_{k=0}^{\infty}\frac{1}{k!}x^k = \sqrt{x}e^x.$$

So we have one solution of the given differential equation. Here we have written the series in terms of elementary functions. However this is not always possible.

40.1.2 Problem 2

Solve the Bessel's equation *of order* p.

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0. \tag{40.1}$$

where $2p$ *is not an integer.*

Solution: We take the following generalized power series

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \tag{40.2}$$

which implies

$$y' = \sum_{m=0}^{\infty} c_m(k+m)x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2}$$

Substitution for y, y', y'' in (40.2) gives

$$x^2 \sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2} + x \sum_{m=0}^{\infty} c_m(k+m)x^{k+m-1} + (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

Combining the first two series we obatin

$$\sum_{m=0}^{\infty} c_m \Big\{(k+m)(k+m-1) + (k+m) - p^2\Big\} x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

Further simplifications leads to

$$\sum_{m=0}^{\infty} c_m(k+m+p)(k+m-p)x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \tag{40.3}$$

Equating the smallest power of x to zero, we get the indicial equation as

$$c_0(k+p)(k-p) = 0, \quad i.e, \quad (k+p)(k-p) = 0, \quad \text{as} \quad c_0 \neq 0.$$

So the roots of indicial equation are $k = p, -p$. Next equating to zero the coefficient of x^{k+1} in (40.3) gives

$$c_1(k+1+p)(k+1-p) = 0, \quad \text{so that} \quad c_1 = 0 \quad \text{for} \quad k = p \quad \text{and} \quad -p.$$

Finally equating to zero the coefficient of x^{k+m} in (40.3) gives

$$c_m(k+m+p)(k+m-p)+c_{m-2}=0$$

$$\Rightarrow \quad c_m=\frac{1}{(k+m+p)(p-k-m)}c_{m-2}.$$

$$\Rightarrow \quad c_m=\frac{1}{(k+m+p)(p-k-m)}c_{m-2}. \tag{40.4}$$

Putting $m=3,5,7,\ldots$ in (40.4) and using $c_1=0$, we find

$$c_1=c_3=c_5=c_7=\ldots=0.$$

Putting $m=2,4,6,\ldots$ in (40.4), we find

$$c_2=\frac{1}{(k+2+p)(p-k-2)}c_0$$

$$c_4=\frac{1}{(k+4+p)(p-k-4)}c_2=\frac{1}{(k+4+p)(p-k-4)(k+2+p)(p-k-2)}c_0$$

and so on. Putting these values in (40.2) and also replacing c_0 by 1, we get

$$y=\left[1+\frac{x^2}{(k+2+p)(p-k-2)}+\frac{x^4}{(k+4+p)(p-k-4)(k+2+p)(p-k-2)}+\ldots\right]$$

Replacing k by p and $-p$ in the above equation gives

$$y_1=x^p\left[1-\frac{x^2}{4(1+p)}+\ldots\right]=x^p\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2^{2k}k!(k+p)(k-1+p)\cdots(2+p)(1+p)}$$

$$y_2=x^{-p}\left[1-\frac{x^2}{4(1-p)}+\ldots\right]=x^{-p}\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2^{2k}k!(k-p)(k-1-p)\cdots(2-p)(1-p)}$$

Therefore when $2p$ is not an integer, we have the general solution to Bessel's equation of order p

$$y=c_1y_1(x)+c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

Remark: *We define the Bessel functions of the first kind Bessel function of the first kind of order p and $-p$ as*

$$J_p(x) = \frac{1}{2^p\Gamma(1+p)} y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$
$$J_{-p}(x) = \frac{1}{2^{-p}\Gamma(1-p)} y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}.$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. When p is not an integer, J_p and J_{-p} are linearly independent. When $2p$ is an integer we obtain

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} \left(\frac{x}{2}\right)^{2k+p}.$$

In this case it turns out that

$$J_p(x) = (-1)^n J_{-p}(x),$$

and so in that case we do not obtain a second linearly independent solution.

40.1.3 Problem 3

Find one series solution of $xy'' + y' + y = 0$.

Solution: The indicial equation is

$$r(r-1) + r = r^2 = 0.$$

This equation has only one root $r = 0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is

$$a_n(r) = (-1)^n \frac{1}{(r+1)^2(r+2)^2 \cdots (r+n)^2}$$

Setting $r = 0$ gives the solution

$$y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2}.$$

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

Raisinghania, M.D. (2009). Advanced Differential Equations. Twelfth Edition. S. Chand & Company Ltd., New Delhi.

Kreyszig, E. (1993). Advanced Engineering Mathematics. Seventh Edition, John Willey & Sons, Inc., New York.

Arfken, G.B. (2001). Mathematical Methods for Physicists. Fifth Edition, Harcourt Academic Press, San Diego.

Grewal, B.S. (2007). Higher Engineering Mathematics. Fourteenth Edition. Khanna Publishilers, New Delhi.

Dubey, R. (2010). Mathematics for Engineers (Volume II). Narosa Publishing House. New Delhi.

Edwards, C.H., Penney, D.E. (2007). Elementary Differential Equations with Boundary Value Problems. Sixth Edition. Pearson Higher Ed, USA.