Lesson 4

Limit, Continuity of Functions of Two Variables

4.1 Introduction

So far we have studied functions of a single (independent) variables. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by the force $(W = F.D)$ and the volume of the rigid circular cylinder ($V = \pi r^2 h$) are both functions of two variables. The volume of a rectangular solid ($V = xyz$) a function of three variables. The notation for a function of two or more variables is similar to that for a function of single variable.

Example 4.1: $z = f(x, y) = x^2 + xy$ (two variables)

Example 4.2: $w = f(x, y, z) = x + 2y - 3z$ (three variable)

A function f of two variables is a rule that assigns a real number $z = f(x, y)$ to each ordered pair (x, y) of real numbers in the domain of f. The range of f is the set of all values of the function: $\{z | z = f(x, y)$ where $(x, y) \in D\}$.

In concrete terms: A function $z = f(x, y)$ is usually just a formula involving the two variables x and y. For every x and y we put in, we get a number z out. The set of all (x, y) we allowed to put into the function is called the domain of the function. Usually the domain is unspecified, and then the domain is the set of all (x, y) we can put into the formula for f and not get square roots of negatives, or division by zero, or some such. i.e.,the domain is usually the set of all (x, y) we can put into the function without getting an undefined expression.

This is the natural domain. The range is simply all the numbers z we can “hit” by putting all (x, y) from the domain into the function.

Example: 4.3: Let $f(x, y) = \sqrt{49 - x^2 - y^2}$. The domain is the disk of radius 7, centre at origin. Now $49 - x^2 - y^2$ will be bigger if x, y ar each smaller. So $f(x, y)$ is biggest when $x = y = 0$. This is $f(0,0) = 7$. Now the smallest value can achieve is 0, when $49 - x^2 - y^2 = 0$ (which happens, for example when $x = 7$ and $y = 0$). If $49 - x^2 - y^2 < 0$, f could not be defined. Hence the range is [0,7].

Definition. The graph of a function f of two variables is the set

$$\{(x, y, z) | z = f(x, y) \ for \ some \ (x, y) \in D\},$$

where D is the domain of f. That is, the graph is the surface $z = f(x, y)$ in 3-dimensional Euclidean Space $\mathbb{R}^3$.

4.1.1 A contour curves or level curves

A contour curve for a function $z = f(x, y)$ is a trace of the surface $z = f(x, y)$ parallel to the xy-plane. That is, let $z = k$ for some number k, and plot $k = f(x, y)$ in the xy-plane.

The domain of a function of two variables f, which is denoted $dom(f)$ from now onwards is the set of all points (x, y) in the xy-plane for which $f(x, y)$ is defined. For example, $A = \{(x, y) | x > y\}$ means that A is the set of points (x, y) such that x is greater than y.

Example 4.4. Determine the domain of $f(x, y) = \ln(y - 2x)$

Solution:

Since the argument of $ln(.)$ must be positive, the domain of f is the set of points (x, y) for which the denominator is not equal to 0. However, $y - 2x > 0$ means that $y > 2x$. In set notation this is written as $dom(f) = \{(x, y)|y > 2x\}$.

Most of the sets in the xy-plane we encounter will be bounded by a closed curve.

As a result, we define an open region to be the set of all points inside of but not including a closed curve, and we define a closed region to be the set of all points inside of and including a closed curve.

Equivalently, a point (p, q) is said to be a boundary point of a set S if any circle centered at (p, q) contains both points inside of and outside of S, and correspondingly, a set S is open if it contains none of its boundary points and closed if it contains all of its boundary points.

Example 4.5. Determine if the domain of the following function is open or closed. $f(x, y) = \sqrt{9 - x^2 - y^2}$

Solution:

To begin with, the quantity $9 - x^2 - y^2$cannot be negative since it is under the square root. Thus, the domain of f is the set of points that satisfy

$$9 - x^2 - y^2 \geq 0 \text{ or } 9 \geq x^2 + y^2.$$

That is, the domain is the set of points (x, y) inside and on the circle of radius 3 centered at the origin, which we write as $dom(f) = \{(x, y)|x^2 + y^2 \leq 9\}$.

Moreover, the domain is a closed region of the xy-plane since it contains the boundary circle of radius 3 centered at the origin.

We say that a region S is connected if any two points in S can be joined by a curve which is contained in S:

4.1.2 Functions of Space and Time

Functions of two variables are important for reasons other than that their graph is a surface. In particular, a function of the form $u(x,t)$ is often interpreted to be a function of x at a given point in time. For example, let's place an xy-coordinate system on a violin whose strings have a length of l, If $u(x,t)$ is considered the displacement of a string above or below a horizontal line at a point x and at a time t, then $y = u(x,t)$ is the shape of the string at a fixed time t.

Likewise, u (x, t) might represent the temperature at a distance $x$from one end of the rod at time t.

4.1.3 Limits and Continuity

Now we will extend the properties of limits and continuity from the familiar function of one variable to the new territory of functions of two or more variables.

Let us recall limit of function of single variable: Let f be a function defined on

an open interval containing a (except possible at a) and let L be a real number.

The statement $\lim_{x\to a} f(x) = L$ means that for given $\varepsilon > 0$, there exists a $\delta > 0$

such that $|f(x) - L| < \varepsilon$, whenever $|x - a| < \delta$.

In less formal language this means that, if the limit holds, then $f(x)$ gets closer

and closer to L as x gets closer and closer to a.

Consider the following limits.

$$\lim_{x\to -2} \frac{x-2}{x^2-4} = \frac{-2-2}{(-2)^2-4} = \frac{-4}{0} \to ?$$

Good job if you saw this as "limit does not exist" indicating a vertical asymptote at $x = -2$.

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\frac{2-2}{(2)^2-4}=\frac{0}{0}\to ?$$

This limit is indeterminate. With some algebraic manipulation, the zero factors could cancel and reveal a real number as a limit. In this case, factoring leads to……

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x+2)(x-2)}$$

$$=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

The limit exists as x approaches 2 even though the function does not exist. In the first case, zero in the denominator led to a vertical asymptote; in the second case the zeros cancelled out and the limit reveals a hole in the graph at $(2,\frac{1}{4})$.

The concept of limits in two dimensions can now be extended to functions of two variables.

Definition 4.1 Let f be a function of two variables defined on an open disc centered at $(x_{0,}y_0)$ i.e., $\{(x,y)|\sqrt{(x-x_0)^2+(y-y_0)^2}<r^2\}$, except polssible at $(x_{0,}y_0)$, and let L be the real numbers Then

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$|f(x,y) - L| < \epsilon$ whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

Graphically for any point $(x,y) \neq (x_0, y_0)$ in the disc with radiusδ, the value $f(x,y)$ lies between $L - \epsilon$ and $L + \epsilon$.

Example 4.6 Let $z = f(x,y) = x^2 + y^2 + 3$.

For the limit of this function to exist at (-1,3), values of z must get closer to 13 as points (x,y) on the xy-plane get closer and closer to (-1,3). $\lim_{(x,y)\to(-1,3)} f(x,y) = 13$. For proof we have to go back to epsilon and delta.

Example 4.7 Verifying the limit by definition $\lim_{(x,y)\to(a,b)} x = a$.

Solution:

We have to show that $|x - a| < \varepsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. Now

$|x - a| \leq \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-a)^2} < \delta$. Let $\delta = \varepsilon$.

Example 4.8. Show that $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$.

Solution:

Now

$$|\frac{5x^2y}{x^2+y^2}| = 5|y|(\frac{x^2}{x^2+y^2}) \leq 5|y|$$

$$\leq 5\sqrt{x^2+y^2} < 5\delta.$$

Put $\delta = \frac{\varepsilon}{5}$, whenever $\sqrt{(x-0)^2+(y-0)^2} < \delta$.

Example 4.9.

Solution:

To show that $|(2x-3y)-(-4)| < \varepsilon$, whenever

$$\sqrt{(x-1)^2+(y-2)^2} < \delta.$$

Now $|2x-3y+4| = |2(x-1)-3(y-2)| \leq 2|x-1| + 3|y-2|$

$< 2\delta + 3\delta = 5\delta$. Set $\delta = \frac{\varepsilon}{5}$.

For a single variable function we have $\lim_{x\to a} f(x)$ has two direction i.e.,

$$\lim_{x\to a^+} f(x) \ and \ \lim_{x\to a^-} f(x).$$

But in case of function of two variables the $\lim_{(x,y)\to(a,b)} f(x,y)$, (x,y) approaches to (a,b) in infinitely many directions.

Example 4.10: Test whether $\lim_{(x,y)\to(0,0)} (\frac{x^2-y^2}{x^2+y^2})^2$ exists.

Solution:

Let $(x,y) \to (0,0)$ on the line $y = mx$. So

$$\lim_{(x,y)\to(0,0)} (\frac{x^2-y^2}{x^2+y^2})^2 = \lim_{x\to 0} (\frac{x^2-m^2x^2}{x^2+m^2x^2})^2$$

$$= (\frac{1-m^2}{1+m^2})^2.$$

As depend on m, so the limit does not exist.

Example 11: Solution: Let $x = r\cos\theta$, $y = r\sin\theta$, $(x,y) \to (0,0)$ implies $r \to 0$. The limit becomes $\lim_{r\to 0} \frac{\sin r^2}{r^2} = \lim_{t\to 0} \frac{\sin t}{t} = 1.$

Definition of Continuity of a Function of Two Variables

A function of two variables is continuous at a point (a,b) in an open region S if $f(a,b)$ is equal to the limit of $f(x,y)$ as (x,y) approaches (a,b). In limit notation:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Give Definition

The function f is continuous in the open region S if f is continuous at every point in S.

The following results are presented without proof. As was the case in functions of one variable, continuity is "user friendly". In other words, if k is a real number and f and g are continuous functions at (a,b) then the functions below are also continuous at (a,b):

$$kf(x,y) = k[f(x,y)], (f \pm g)(x,y) = f(x,y) \pm g(x,y),$$

$$(fg)(x,y) = f(x,y)g(x,y), (\frac{f}{g})(x,y)$$

$$= \frac{f(x,y)}{g(x,y)} \ if \ \ g(a,b) \neq 0.$$

The conclusions indicate that arithmetic combinations of continuous functions are also continuous —that polynomial and rational functions are continuous on their domains.

Finally, the following result asserts that the composition of continuous functions are also continuous. If f is continuous at (a, b) and g is continuous at $f(a, b)$,

then the composition function $(gof)(x, y) = g(f(x, y))$ is continuous at (a, b)

and

$$\lim_{(x,y)\to(a,b)} g(f(x, y)) = g(f(a, b)).$$

Example 4.12 Find the limit and discuss the continuity of the function $\lim_{(x,y)\to(1,2)} \frac{x}{\sqrt{2x+y}}$

Solution:

$\lim_{(x,y)\to(1,2)} \frac{x}{\sqrt{2x+y}} = \frac{1}{\sqrt{2(1)+2}} = \frac{1}{2}$. The function will be continuous when

$2x + y > 0$.

Example 4.13. Using ε and δ show that the function $f(x, y) = x^3 - 3xy^2$ is

continuous at origin.

Solution:

Set $x = r\cos\theta$ and $y = r\sin\theta$ (θ is fixed). Then

$|f(x,y)| = r^3|\cos^3\theta - 3\cos\theta\sin^2\theta| < 4r^3$. Take $r = \sqrt{x^2+y^2} < \delta = (\frac{\varepsilon}{4})^{\frac{1}{3}}$.

Example 4.14. Is it possible to define $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ at $(0,0)$ so that $f(x,y)$ is continuous?

Solution:

Note that

$$|\frac{x^3+y^3}{x^2+y^2}| \leq \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} = \frac{x^2|x|}{x^2+y^2} + \frac{y^2|y|}{x^2+y^2}$$

$$\leq |x| + |y| \leq 2\sqrt{x^2+y^2} = \varepsilon$$

where $\sqrt{x^2+y^2} < \delta$ and $\delta = \frac{\varepsilon}{2}$. If we define $f(0,0) = 0$, $f(x,y)$ is continuous every where.

Example 4.15. Show that the function $\frac{\sin xy}{\sqrt{x^2+y^2}}$ is continuous if we define $f(0,0) = 0$.

Solution:

Discontinuity possible only at $(0,0)$. Note with $x = r\cos\theta$ and $y = r\sin\theta$, from $|\sin\alpha| < |\alpha|$ for small α , that $|\frac{\sin xy}{\sqrt{x^2+y^2}}| < r$; hence limit at $(0,0)$ exists and is 0.

Property 1: If a function $f(x, y)$ is defined and continuous in a closed and bounded domain D, then there will be at least one point (x^*, y^*) in D such that

$$f(x^*, y^*) \geq f(x, y).$$

And at least one point $f(x_*, y_*) \in D$ such that

$$f(x_*, y_*) \leq f(x, y).$$

We call $f(x^*, y^*) = M$ as the maximum value of the function and $f(x_*, y_*) = m$ is the minimum value of the function. This result states that a function which is continuous on a closed and bounded domain D has a maximum and minimum.

Property 2: If $f(x, y)$ has both maximum and minimum M and m respectively, let $m < \mu < M$, then $\exists\ (x_1, y_1) \in D$ such that $f(x_1, y_1) = \mu$.

Corollary to property 2.

If a function $f(x, y)$ is continuous in a closed and bounded domain D and assumes both positive and negative values, then there will be a point inside the domain at which the $f(x, y)$ vanishes.

Questions: Answer the following questions.

1. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$, if it exists.

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2+1} = 0$

3. Prove that $\lim_{(x,y)\to(1,2)} 2x - 3y = -4$.

4. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

5. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$, if it exists.

6. Test for continuity $(a)\ f(x,y) = \frac{x-2y}{x^2+y^2}$ $(b)\ g(x,y) = \frac{2}{y-x^2}$

7. Find the $\lim_{(x,y)\to(0,1)} \frac{\sin^{-1}\left(\frac{x}{y}\right)}{1+xy}$ and discuss the continuity of the function $\frac{\sin^{-1}\left(\frac{x}{y}\right)}{1+xy}$ at $0,1)$.

8. Find the $\lim_{(x,y)\to(0,0)} \frac{-1}{2}\ln(x^2+y^2)$,

and discuss the continuity of the function $\frac{-1}{2}\ln(x^2+y^2)$ at $(0,0)$.

Example 1: Let $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$ for $(x,y) = (0,0)$. Is it continuous at $(0,0)$ or can we make continuous by redefining $f(0,0)$? (Hint: not possible)

Example 2: Is it possible to extend $f(x,y) = \frac{x+y}{x^2+y^2}$ to the origin so that the resulting function is continuous? (Hint: not possible)

Keywords: Limit, Continuity, Maximum and Minimum values.

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Suggested Readings

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