Lesson 39

Series Solutions about a Regular Singular Point

39.1 Introduction

In this lesson we discuss series solution about a singular point. In particular, the power series method discussed in last lessons will be generalized. The generalized power series method is also known as Frobenius method.

Let us consider a simple first order differential equation $2xy' - y = 0$ and try to apply the power series method discussed in the last lessons. Note that $x = 0$ is a singular point. If we plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

into the given differential equation, we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k)\, x^k. \end{aligned}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\, a_k$ for $k = 1, 2, 3, \ldots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

39.2 Frobenius Method

Consider the differential equation of the form $y'' + p(x)y' + q(x)y = 0$. Note that $xp(x)$ and $x^2 q(x)$ are analytic at $x = 0$. We try a series solution of the from

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = x^r (c_0 + c_1 x + c_2 x^2 + \ldots), \text{ where } c_0 \neq 0$$

The derivative of y with respect to x are given by

$$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

Also, we can write power series corresponding to $xp(x)$ and $x^2q(x)$ as

$$xp(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } x^2q(x) = \sum_{n=0}^{\infty} b_n x^n$$

The given differential equation can be rewritten as

$$y'' + \frac{xp(x)}{x}y' + \frac{x^2q(x)}{x^2}y = 0$$

Substituting all values of y, y', y'', $xp(x)$ and $x^2q(x)$ series into the above differential equation we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n-1} \times \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} b_n x^{n-2} \times \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Multiplying by x^2 we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n} \times \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} b_n x^{n} \times \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

We can now equate coefficients of various powers of x to zero to form a system of equations involving unknown coefficients c_n. Equating the coefficient of x^r we obtain

$$[r(r-1) + a_0 r + b_0]c_0 = 0$$

Since $c_0 \neq 0$, we obtain

$$r^2 + (a_0 - 1)r + b_0 = 0 \tag{39.1}$$

The above quadratic equation is known as the *indicial equation* of the given differential equation. The general solution of the given differential equation depends on the roots of the indicial equation. There are three possible general cases:

39.2.1 Case I: The indicial equation has two real roots which do not differ by an integer

Let r_1 and r_2 are the roots of the indicial equation. Then the two linearly independent solution will follow from

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} c_n z^n \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} \overline{c}_n z^n$$

where $c_0, c_1, \ldots$ are coefficients corresponding to $r = r_1$ and $\overline{c}_0, \overline{c}_1, \ldots$ are coefficients corresponding to $r = r_2$. The general solution will be of the form $y = ay_1 + by_2$, where a and b are arbitrary coefficients.

39.2.2 Case II: The indicial equation has a doubled root

If the indicial equation has a doubled root r, then we find one solution

$$y_1 = x^r \sum_{k=0}^{\infty} a_k x^k,$$

and then obtain another solution by plugging

$$y_2 = x^r \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into the given equation and solving for the constants b_k.

39.2.3 Case III: The indicial equation has two real roots which differ by an integer

If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into the given equation and solve for the constants b_k and C.

Remark 1: *Note that the case-I also includes complex numbers because in that case $r_1 - r_2$ will be a complex number which cannot be equal to a real integer.*

Remark 2: *Note that the mai idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or the method of variation of parameters to obtain a second solution.*

39.3 Working Rules

Now we summarize the working steps of the Frobenius method:

1. We seek a Frobenius-type solution of the form $y = \sum_{k=0}^{\infty} a_k x^{k+r}$.
2. We plug this y into the given differential equation.
3. The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.
4. If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then find two linearly independent solutions according to Case-I.
5. If the indicial equation has a doubled root r, or the indicial equation has two real roots such that $r_1 - r_2$ is an integer then follow Case-II or Case-III accordingly.

39.3.1 Example

Find the power series solutions about $x = 0$ of

$$4xy'' + 2y' + y = 0$$

Solution: Clearly, $x = 0$ is a regular singular point. Comparing with $y'' + p(x)y' + q(x)y = 0$ we have $xp(x) = 1/2$ and $x^2 q(x) = x/4$. We substitute Frobenius series

$$y = x^r \sum_{n=0}^{\infty} c_n x^n \tag{39.2}$$

into the differential equation to get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \frac{1}{2x}\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \frac{1}{4x}\sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

Multiplying by x^2 we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \frac{1}{2}\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \frac{1}{4}\sum_{n=0}^{\infty}c_n x^{n+r+1} = 0 \qquad (39.3)$$

Equating coefficients of x^r to zero and noting $c_0 \neq 0$ we obtain indicial equation

$$r(r-1) + \frac{1}{2}r = 0$$

which has roots $r = 1/2, 0$. These roots are unequal and do not differ by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{n+r} in Equation (39.3) and obtain

$$(n+r)(n+r-1)c_n + \frac{1}{2}(n+r)c_n + \frac{1}{4}c_{n-1} = 0$$

Corresponding to $r = 1/2$ we get

$$(4n^2+2n)c_n + c_{n-1} = 0 \;\;\Rightarrow\;\; c_n = -\frac{c_{n-1}}{2n(2n+1)} \;\;\Rightarrow\;\; c_n = -c_0\frac{(-1)^n}{(2n+1)!}$$

Substituting these values in (39.2), we get one solution as

$$y_1 = c_0\sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n = c_0\left(\sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} + \dots\right) = \sin\sqrt{z}$$

To obtain the second solution we use $r = 0$ to get

$$(4n^2-2n)c_n + c_{n-1} = 0 \;\;\Rightarrow\;\; c_n = -\frac{c_{n-1}}{2n(2n-1)} \;\;\Rightarrow\;\; c_n = \frac{(-1)^n}{(2n)!}$$

Hence the second solution is

$$y_2 = c_0\sum_{n=0}^{\infty}x^n = \cos(\sqrt{z})$$

The general solution is given as

$$y = b\cos(\sqrt{z}) + b\cos(\sqrt{z})$$

where a and b are arbitrary constants.

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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