# Lesson 38

# Series Solution about an Ordinary Point (Cont.)

In the last lesson we have discussed series solution of the homogeneous differential equations. In this lesson we demonstrate the method by using a couple of basic examples. For demonstration we take first example of a differential equation with constant coefficients and then some more involved examples will be discussed.

## 38.1 Example Problems

### 38.1.1 Problem 1

*Determine a series solution to* $y'' - y = 0$.

**Solution:** Suppose that the series solution is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x_n$$

Differentiating $y$, we have

$$y'(x) = \sum_{n=1}^{\infty} n c_n x_{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x_{n-2}$$

Substituting these into the differential equation, we have

$$\sum_{n=2}^{\infty} n(n-1) c_n x_{n-2} - \sum_{n=0}^{\infty} c_n x_n = 0$$

Re-indexing the first sum

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x_n - \sum_{n=0}^{\infty} c_n x_n = 0$$

This implies

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - c_n x_n\right] x_n = 0$$

Since the series is always equal to $0$ then each coefficient must be zero. Thus we have

$$(n+2)(n+1)c_{n+2} - c_n = 0 \tag{38.1}$$

This can be rewritten in the form of recurrence relation as

$$c_{n+2} = \frac{c_n}{(n+2)(n+1)} \tag{38.2}$$

Putting $n = 0, 1, 2\ldots$, we get

$$c_2 = \frac{c_0}{2!}, \quad c_3 = \frac{c_1}{3!}, \quad c_4 = \frac{c_0}{4!}, \quad c_5 = \frac{c_1}{5!}, \ldots$$

In general, we have

$$c_{2k} = \frac{c_0}{(2k)!}, \quad c_{2k+1} = \frac{c_1}{(2k+1)!} \ldots \text{ for } k = 1, 2, \ldots.$$

Putting these values into the series and collecting the $c_0$ and $c_1$ terms we get

$$y(x) = c_0\left(1 + \frac{x^2}{2!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots\right) + c_1\left(x + \frac{x^3}{3!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots\right)$$

This can be further rewritten in summation form as

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

This is the desired series solution. It should be noted that this series solution can be rewritten into the form of well known solution $y(x) = \overline{c}_1 e^x + \overline{c}_2 e^{-x}$ of the given differential equation as

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = \overline{c}_1\left(1 + x + \frac{x^2}{2!} + \ldots\right) + \overline{c}_2\left(1 - x + \frac{x^2}{2!} + \ldots\right)$$

This can be rewritten as

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = (\overline{c}_1 + \overline{c}_2)\left(1 + \frac{x^2}{2!} + \ldots\right) + (\overline{c}_1 - \overline{c}_2)\left(x + \frac{x^3}{3!} + \ldots\right)$$

Denoting $(\overline{c}_1 + \overline{c}_2) =: c_0$ and $(\overline{c}_1 - \overline{c}_2) =: c_1$ we get

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

This proves that both representations are equivalent.

### 38.1.2 Problem 2

*Find the series solution, about $x = 0$, of the equation $(1-x)^2 y'' - 2y = 0$ in powers of $x$.*

**Solution:** Since $x = 0$ is an ordinary point and we can therefore get two linearly independent solution by substituting

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

After substitution we get

$$(1 - 2x + x^2) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2 \sum_{n=0}^{\infty} c_n x^n = 0,$$

which leads to

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) c_n x^n - 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

In order to write the series in terms the coefficients of $x^n$ we shift the summation index as

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 2 \sum_{n=1}^{\infty} n(n+1) c_{n+1} x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^n - 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

The sum in second and third series can also start from 0 without changing the series. This leads to

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) c_{n+2} - 2n(n+1) c_{n+1} + n(n-1) c_n - 2c_n \right] x^n = 0$$

This can be further simplified as

$$\sum_{n=0}^{\infty} (n+1) \left[ (n+2) c_{n+2} - 2n c_{n+1} + (n-2) c_n \right] x^n = 0$$

Equating the coefficients we obtain the recurrence relation

$$(n+2) c_{n+2} - 2n c_{n+1} + (n-2) c_n = 0.$$

Putting $n = 0, 1, 2, \ldots$ we get

$$c_2 = c_0, \quad c_3 = \frac{1}{3} (2c_0 + c_1) =: c, \quad c_4 = c, \quad c_5 = c \ldots$$

Hence the series solution becomes

$$y = c_0 + c_1 x + c_0 x^2 + c \sum_{n=3}^{\infty} x^n.$$

### 38.1.3 Problem 3

*Find the power series solution of the equation* $(x^2+1)y''+xy'-xy=0$ *in powers of* $x$ *(i.e. about* $x=0$*).*

**Solution:** Clearly $x=0$ is an ordinary point of the given differential equation. Therefore, to find the series solution, we take power series

$$y=c_0+c_1x+c_2x^2+c_3x^3+\ldots=\sum_{n=0}^{\infty}c_nx^n. \tag{38.3}$$

Differentiating twice in succession, (38.3) gives

$$y'=\sum_{n=1}^{\infty}nc_nx^{n-1} \quad \text{and} \quad y''=\sum_{n=1}^{\infty}n(n-1)c_nx^{n-2} \tag{38.4}$$

Putting the above value of $y, y'$ and $y''$ in the given differential equation, we obtain

$$(x^2+1)\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+x\sum_{n=1}^{\infty}nc_nx^{n-1}-x\sum_{n=0}^{\infty}nc_nx^n=0$$

$$\Rightarrow \quad \sum_{n=2}^{\infty}n(n-1)c_nx^n+\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-\sum_{n=1}^{\infty}nc_nx^n-\sum_{n=0}^{\infty}c_nx^{n+1}=0$$

This leads to

$$\sum_{n=2}^{\infty}n(n-1)c_nx^n+\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n+\sum_{n=1}^{\infty}nc_nx^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$$

Finally we have the identity

$$2c_2+(6c_3+c_1-c_0)x+\sum_{n=2}^{\infty}[n(n-1)c_n+(n+2)(n+1)c_{n+2}+nc_n-c_{n-1}]x^n=0.$$

Equating the constant term and the coefficients of various powers of $x$, we get

$$c_2=0,\ 6c_3+C_1-c_0=0 \text{ so that } c_3=(c_0-c_1)/6$$

and the recurrence relation

$$c_{n+2}=\frac{c_{n-1}-n^2c_n}{(n+1)(n+2)},\ \text{for all } n\geq 2. \tag{38.5}$$

Putting $n = 2$ in (38.5), $c_4 = (1/12)c_1$, as $c_2 = 0$.

Putting $n = 3$ in (38.5), $c_5 = -\frac{9c_3}{(20)} = -\frac{3}{40}(c_0 - c_1)$

Putting the above values of $c_2, c_3, c_4, c_5, \ldots$ ets. in (38.3), we have

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots \infty$$

$$\Rightarrow \quad y = c_0 + c_1 x + (1/6)(c_0 - c_1)x^3 + (1/12)c_1 x^4 - (3/40)(c_0 - c_1)x^5 + \ldots \infty$$

This can be rewritten as

$$y = c_0\Big(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \ldots\Big) + c_1\Big(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \ldots\Big),$$

which is the required solution near $x = 0$, where $c_0$ and $c_1$ are arbitrary constants.

### 38.1.4 Problem 4

*Find the power series solution of the initial value problem* $xy'' + y' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$ *in powers of* $(x - 1)$.

**Solution:** Since $x = 1$ is an ordinary point of the given differential equation, we find series solution

$$y = \sum_{n=0}^{\infty} c_n (x-1)^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n c_n (x-1)^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1)c_n (x-1)^{n-2} \quad (38.6)$$

Substituting $y$ and $y'$ in the given differential equation we obtain

$$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)c_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n c_n (x-1)^{n-1} + 2\sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

This leads to

$$\sum_{n=2}^{\infty} n(n-1)c_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n c_n (x-1)^{n-1} + 2\sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

Shifting summation index of the first three terms we get

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}(x-1)^n + \sum_{n=0}^{\infty}[(n+1)(n+2)c_{n+2} + (n+1)c_{n+1} + 2c_n](x-1)^n = 0$$

Equating the coefficients to zero we get

$$2c_2 + c_1 + c_0 = 0 \quad \Rightarrow \quad c_2 = -\frac{c_1 + c_0}{2}$$

$$c_{n+2} = -\frac{(n+1)^2 c_{n+1} + 2c_n}{(n+1)(n+2)}, \text{ for all } n \geq 1$$

Using initial conditions in Equation (38.6) we get $c_0 = 1$ and $c_1 = 2$. Using these values we obtain

$$c_2 = -2, \quad c_3 = \frac{2}{3}, \quad c_4 = -\frac{1}{6}, \quad c_5 = \frac{1}{15}, \ldots$$

Putting these constants in series we get the desired solution as

$$y = 1 + 2(x-1) - 2(x-1)^2 + (2/3)(x-1)^3 - (1/6)(x-1)^4 + (1/15)(x-1)^5 + \ldots$$

## Suggested Readings

Waltman, P. (2004). A Second Course in Elementary Differential Equations. Dover Publications, Inc. New York.

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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