Lesson 37

Series Solutions about an Ordinary Point

37.1 Introduction

If we can't find a solution to a differential equations in a form of nice functions, we can still look for a series representation of the solution. Series solutions are very useful because if we know that the series converges, we can approximate the solution as closely as we want. In this lesson we describe series solutions of solving second order linear homogeneous differential equations with variables coefficients. Series solution can be used in conjunction with variation of parameters to solve linear nonhomogeneous equations. For simplicity, we shall be dealing mainly with polynomial coefficients. Here we consider the second order homogeneous equation of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{37.1}$$

where P, Q and R are polynomials or analytic functions in general. Many problems in mathematical physics leads to equations of the form (37.1) having polynomial coefficients; for example, the Bessel equation

$$x^2y'' + xy' + (x^2a^2)y = 0,$$

where a is a constant, and the Legendre equation

$$(1-x)^2y'' - 2xy' + c(c+1)y = 0$$

where c is a constant.

37.2 Useful Definitions

Here we provide some definitions which will be very useful for finding series solution of the differential equations.

37.2.1 Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0) \tag{37.2}$$

exists and converges to $f(x)$ for all x in the interval of convergence of (37.2).

37.2.2 Ordinary Points

A point $x = x_0$ is called an ordinary point of the Equation (37.1) if P, Q, and R are polynomials that do not have any common factors, then a point x_0 is called an ordinary point if $P(x_0) \neq 0$. A point x_1 where $P(x_1) = 0$ is called a singular point. If any of P, Q, or R is not a polynomial, then we call x_0 an ordinary point if $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic about x_0.

It is often useful to rewrite Equation (37.1) as

$$y'' + p(x)y' + q(x)y = 0 \tag{37.3}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. The Equation (37.3) is called equivalent normalized form of the Equation (37.1).

37.2.3 Singular Points

If the point $x = x_0$ is not an ordinary point of the differential Equation (37.1) or (37.3), then it is called a singular point of the differential equation of (37.3). There are two types of singular points: (i) regular singular points, and (ii) irregular singular points. A singular point $x = x_0$ of the differential Equation (37.3) is called a regular singular point of the differential Equation (37.3) if both

$$(x - x_0)p(x) \text{ and } (x - x_0)^2 q(x)$$

are analytic at $x = x_0$. A singular point, which is not regular is called an irregular singular point.

37.3 Example Problems

37.3.1 Problem 1

Show that $x = 0$ is an ordinary point of $(x^2 - 1)y'' + xy' - y = 0$, but $x = 1$ is a regular singular point.

Solution: Writing the given equation in normalized form

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)(x+1)}\frac{dy}{dx} - \frac{1}{(x-1)(x+1)}y = 0. \tag{37.4}$$

Comparing (37.4) with the standard equation $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = x/(x-1)(x+1) \text{ and } q(x) = -1/(x-1)(x+1).$$

Since both $p(x)$ and $q(x)$ are analytic at $x = 0$, the point $x = 0$ is an ordinary point of the given Equation (37.4). Further note that both $p(x)$ and $q(x)$ are not analytic at $x = 1$, thus $x = 1$ is not an ordinary point and so $x = 1$ is a singular point. Also

$$(x-1)P(x) = x/(x+1) \text{ and } (x-1)^2Q(x) = -(x-1)/(x+1)$$

show that both $(x-1)P(x)$ and $(x-1)^2Q(x)$ are analytic at $x = 1$. Therefore $x = 1$ is a regular singular point.

37.3.2 Problem 2

Determine whether the point $x = 0$ is an ordinary point or regular point of the differential equation

$$xy'' + \sin(x)y + x^2y = 0$$

Solution: Comparing with the normalized equation we get

$$p(x) = \frac{\sin x}{x} \text{ and } q(x) = x$$

Since $p(x)$ and $q(x)$ both are analytic at $x = 0$, the point $x = 0$ is an ordinary point. This example shows that singular point does not always occur where $P(x) = 0$.

37.3.3 Problem 3

Discuss the singular points of the differential equation

$$x^2(x-2)^2y'' + (x-2)y' + 3x^2y = 0.$$

Solution: Clearly the function

$$p(x) = \frac{1}{(x^2(x-2))}$$

is not analytic at $x = 0$ and $x = 2$. Also the function

$$q(x) = \frac{3}{((x-2)^2)}$$

is not analytic at $x = 2$. Hence both $x = 0$ and $x = 2$ are singular point of the differential equations. At $x = 0$ we have

$$xp(x) = \frac{1}{(x(x-2))} \quad \text{and} \quad x^2q(x) = \frac{3x^2}{(x-2)^2}$$

Note that $x^2q(x)$ is non-singular at $x = 0$ but $xp(x)$ is not analytic at this point. Hence $x = 0$ is an irregular singular point. At $x = 2$ we have

$$(x-2)p(x) = \frac{1}{x^2} \quad \text{and} \quad (x-2)^2q(x) = 2$$

Both functions are analytic at $x = 2$ and hence $x = 2$ is a regular singular point.

37.4 Brief Overview of Power Series

A power series about a point x_0 is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n$$

where x is a variable and c_n are constants, called coefficients of the series. There are three possibilities about the convergence of a power series. The series may converge only at $x = 0$ or it may converge for all values of x. If this is not the case then a definite positive number R exists such that the given series converges for every $|x - x_0| < R$ and diverges for every $|x-x_0| > R$. Such a number is known as the radius of convergence and $]x_0 - R, x_0 + R[$, the interval of convergence, of the given series.

Among several formulas for determining convergence of the power series, ratio test is most common and simple to use. Given a power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ we compute

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|,$$

then the series is convergence for $|x-x_0| < R$ and divergent $|x-x_0| > R$.

37.4.1 Example

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x+1)^n}{n2^n}$$

Solution: Ratio test gives

$$\lim_{n\to\infty}\left|\frac{n2^n}{(n+1)2^{n+1}}\right| = \frac{1}{2}$$

Hence the radius of convergence of the power series is $R = 2$ and the interval of convergence is $-3 < x < 1$. The convergence at the end points $x = -3$ and $x = 1$ needs to be checked separately.

37.5 Power Series Solution near Ordinary Point

Let the given equation be

$$y'' + p(x)y' + q(x)y = 0 \tag{37.5}$$

If $x = x_0$ is an ordinary point of (37.5), then (37.5) has two non-trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n(x-x_0)^n \tag{37.6}$$

and these power series converge in some interval of convergence $|x-x_0| < R$, (where R is the radius of convergence of (37.6)) about x_0.

To find series solutions we suppose that we have a series representation,

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n \tag{37.7}$$

and then to find out coefficients C_n we need to differentiate (37.7) and plug in the derivatives into the Equation (37.6). Once we have the appropriate coefficients, we call (37.7) the series solution to (37.5) near $x = x_0$. More precisely, differentiating twice, the Equation (37.7) yields

$$y' = \sum_{n=0}^{\infty} nC_n (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1)C_n (x - x_0)^{n-2} \tag{37.8}$$

Substituting the above values of y, y' and y'' in (37.5), we obtain

$$A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \ldots + A_n(x - x_0)^n + \ldots = 0, \tag{37.9}$$

where the coefficients $A_0, A_1, A_2 \ldots$ etc. are now some functions of the coefficients $C_0, C_1, C_2, \ldots$ etc. Since the Equation (37.9) is an identity, all the coefficients $A_0, A_1, A_2 \ldots$ of (37.9) must be zero, i.e.,

$$A_0 = 0, A_1 = 0, A_2 = 0, \ldots, A_n = 0 \tag{37.10}$$

Solving Equation (37.10), we obtain the coefficients of (37.7) in terms of C_0 and C_1. Substituting these coefficients in (37.7), we obtain the required series solution of (37.5) in power of $(x - x_0)$.

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

Raisinghania, M.D. (2009). Advanced Differential Equations. Twelfth Edition. S. Chand & Company Ltd., New Delhi.

Grewal, B.S. (2007). Higher Engineering Mathematics. Fourteenth Edition. Khanna Publishilers, New Delhi.

Edwards, C.H., Penney, D.E. (2007). Elementary Differential Equations with Boundary Value Problems. Sixth Edition. Pearson Higher Ed, USA.