Lesson 36

Methods for Solving Simultaneous Ordinary Differential Equations

In this lesson we shall consider systems of simultaneous linear differential equations which contain a single independent variable and two or more dependent variables. We will consider two different techniques, mainly the method of elimination and the method of differentiation, for solving system of differential equations.

36.1 Simultaneous Ordinary Linear Differential Equations

Let x and y be the dependent and t be the independent variable. Thus, in such equations there occur differential coefficients of x, y with respect to t. Let $D = d/dt$, then such equations can be put into the form

$$f_1(D)x + f_2(D)y = T_1 \tag{36.1}$$

$$g_1(D)x + g_2(D)y = T_2 \tag{36.2}$$

where T_1 and T_2 are functions of the independent variable t and $f_1(D), f_2(D), g_1(D),$ and $g_2(D)$ are all rational integral functions of D with constant coefficients. In general, the number of equations will be equal to the number of dependent variables, i.e., if there are n dependent variables there will be n equations.

36.2 Method of Elimination

In order to eliminate y between equations (36.1) and (36.2), operating on both sides of (36.1) by $g_2(D)$ and on both sides of (36.2) by $f_2(D)$ and subtracting, we get

$$\left(f_1(D)g_2(D) - g_1(D)f_2(D)\right)x = g_2(D)T_1 - f_2(D)T_2 \tag{36.3}$$

This is a linear differential equation with constant coefficients in x and t and can be solved to give the value of x in terms of t. Substituting this value of x in either (36.1) or (36.2), we get the value of y in terms of t.

Remark 1: *The above Equations* (36.1) *and* (36.2) *can be also solved by first eliminating* x *between them and solving the resulting equation to get* y *in terms of* t*. Substituting this value of* y *in either* (36.1) *or* (36.2)*, we get the value of* x *in terms of* t*.*

Remark 2: *In the general solutions of* (36.1) *and* (36.2) *the number of arbitrary constants will be equal to the sum of the orders of the equations* (36.1) *and* (36.2).

36.3 Example Problems

36.3.1 Problem 1

Solve the simultaneous equations

$$\frac{dx}{dt} - 7x + y = 0 \tag{36.4}$$

$$\frac{dy}{dt} - 2x - 5y = 0 \tag{36.5}$$

Solution: Writing D for d/dt, the given equations can be rewritten in the following symbolic form as

$$(D-7)x + y = 0 \tag{36.6}$$

$$-2x + (D-5)y = 0 \tag{36.7}$$

Now, we eliminate x by multiplying Equation (36.6) by 2 and operating (36.7) by $(D-7)$ as follows

$$2(D-7)x + 2y = 0 \tag{36.8}$$

$$-2(D-7)x + (D-7)(D-5)y = 0 \tag{36.9}$$

Adding (36.8) and (36.9), we get

$$[(D-7)(D-5)+2]y = 0$$

or

$$(D^2 - 12D + 37)y = 0$$

This is a linear equation with constants coefficients. Its auxiliary equation is

$$(m^2 - 12m + 37) = 0$$

The roots of the auxiliary equation are $m = 6 \pm i$. Therefore, we get the general solution for the variable y as

$$y = e^{6t}(c_1 \cos t + c_2 \sin t), \tag{36.10}$$

where c_1 and c_2 being arbitrary constants. We now find x by using Equation (36.7). Now from (36.10), differentiating w.r.t. t, we get

$$Dy = 6e^{6t}[(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)],$$

or on simplifications we obtain

$$Dy = 6e^{6t}[(6c_1 + c_2) \cos t + (-c_1 + 6c_2) \sin t] \tag{36.11}$$

Now, substituting y and Dy in the Equation (36.7), we get

$$x = (1/2) \times e^{6t}[(c_1 + c_2) \cos t + (-c_1 + c_2) \sin t] \tag{36.12}$$

Thus, equations (36.10) and (36.12) give the desired general solution.

36.3.2 Problem 2

Solve the linear system of differential equations

$$D^2y - y + 5Dv = x \tag{36.13}$$

$$2Dy - D^2v + 4v = 2 \tag{36.14}$$

Solution: Multiplying (36.13) by $2D$ and (36.14) by $(D^2 - 1)$ and then subtracting (36.14) from the Equation (36.13) we obtain

$$[10D^2 + (D^2 - 1)(D^2 - 4)]v = 2Dx - (D^2 - 1)2$$

or

$$(D^4 + 5D^2 + 4)v = 4 \tag{36.15}$$

This is a linear differential equations with constant coefficients whose solution can easily be found. The characteristic equation of the corresponding homogeneous equation is

$$m^4 + 5m^2 + 4 = 0 \quad \Rightarrow \quad (m^2+1)(m^2+4) = 0 \quad \Rightarrow \quad m = \pm i, \pm 2i$$

The complimentary function is

$$\text{C.F.} = c_1 \cos x + +c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^4 + 5D^2 + 4} 4e^{0x} = 1$$

We write the general solution for v as

$$v = 1 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x \tag{36.16}$$

Now we find an equation giving y in terms of v. This can be done by eliminating from the equations (36.13) and (36.14) those terms which involve derivatives of y. So multiplying Equation (36.13) by 2 and Equation (36.14) by D we get

$$(2D^2 - 2)y + 10Dv = 2x \tag{36.17}$$

$$2D^2 y - (D^3 - 4D)v = 0 \tag{36.18}$$

Subtracting (36.17) from (36.18) we get

$$2y - D^3 v - 6Dv = -2x \tag{36.19}$$

or

$$y = -x + \frac{1}{2} D^3 v + 3Dv \tag{36.20}$$

Substitute v from (36.16) into the Equation (36.21) to obtain the expression for y as

$$y = -x - \frac{5}{2} c_1 \cos x + \frac{5}{2} c_2 \cos x + 2c_4 \cos 4x - 2c_3 \sin 2x, \tag{36.21}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

36.4 Method of Differentiation

Sometimes, x and y can be eliminated if we differentiate (36.1) or (36.2). For example, assume that the given equations (36.1) and (36.2) relates four quantities $x, y, dx/dt$ and dy/dt. Differentiating (36.1) and (36.2) with respect to t, we obtain four equations containing $x, dx/dt, d^2x/dt^2, y, dy/dt$ and d^2y/dt^2. Eliminating three quantities $y, dy/dt$ and d^2y/dt^2 from these four equations, y is eliminated and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get value of x in terms of t. Substituting this value of x in either (36.1) or (36.2), we get value of y in terms of t. The technique will be illustrated by the following example.

36.4.1 Example

Determine the general solutions for x and y for

$$\frac{dx}{dt} - y = t$$
$$\frac{dy}{dt} + x = 1$$

Solution: Writing D for d/dt, the given equations become

$$Dx - y = t \tag{36.22}$$
$$x + Dy = 1 \tag{36.23}$$

Differentiating the equation Equation (36.22) w.r.t. t we get

$$D^2x - Dy = 1 \tag{36.24}$$

Now we can eliminate y by adding equations (36.24) and (36.23) to get

$$(D^2 + 1)x = 2 \tag{36.25}$$

The auxiliary equation of the above differential equation is $m^2 + 1 = 0$ and therefore the general solution of the homogeneous equation is

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

where c_1 and c_2 are arbitrary constants. The particular integral is

$$\text{P.I} = \frac{1}{D^2+1}2 = (1+D^2)^{-1}2 = (1-D^2+...)2 = 2$$

Hence, the general solution of (36.25) is

$$x = c_1 \cos t + c_2 \sin t + 2 \tag{36.26}$$

From Equation (36.22), we get

$$y = c_2 \cos t - c_1 \sin t - t \tag{36.27}$$

Thus, the required solution is given by (36.26) and (36.27).

Suggested Readings

Waltman, P. (2004). A Second Course in Elementary Differential Equations. Dover Publications, Inc. New York.

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

Dubey, R. (2010). Mathematics for Engineers (Volume II). Narosa Publishing House. New Delhi.

McQuarrie, D.A. (2009). Mathematical Methods for Scientist and Engineers. First Indian Edition. Viva Books Pvt. Ltd. New Delhi.

Raisinghania, M.D. (2005). Ordinary & Partial Differential Equation. Eighth Edition. S. Chand & Company Ltd., New Delhi.

Kreyszig, E. (1993). Advanced Engineering Mathematics. Seventh Edition, John Willey & Sons, Inc., New York.

Grewal, B.S. (2007). Higher Engineering Mathematics. Fourteenth Edition. Khanna Publishilers, New Delhi.

Piskunov, N. (1996). Differential and Integral Calculus (Volume - 2). First Edition. CBS Publisher, Moscow.