Lesson 35

Equations Reducible to Linear Differential Equations with Constant Coefficients

In this lesson we shall study two special forms of linear equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by a suitable substitution. Those special forms which we study here are called Cauchy-Euler homogeneous linear differential equations and Legendre's homogeneous linear differential equations.

35.1 Cauchy-Euler Homogeneous Linear Differential Equation

A linear differential equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_ny=F(x), \tag{35.1}$$

where $a_1, a_2, \ldots, a_n$ are constants and F is either a constant or a function of x only, is called Cauchy-Euler homogeneous linear differential equation. Note that the index of x and order of derivative is same in each term of such equations.

Using the symbols $D(=d/dx), D^2(=d^2/dx^2), \ldots, D^n(=d^n/dx^n)$, the Equation (35.1) becomes

$$(a_0x^nD^n+a_1x^{n-1}D^{n-1}+a_2x^{n-2}D^{n-2}+...+a_n)y=F(x) \tag{35.2}$$

The above equation can be reduced to linear differential equation with constant coefficients by substituting

$$x=e^z, \quad \text{or} \quad \ln x = z, \quad \text{so that} \quad \frac{dz}{dx}=\frac{1}{x} \tag{35.3}$$

Using chain rule for differentiation we obtain

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

Defining $\frac{d}{dz} =: D_1$, we have

$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \Leftrightarrow \quad xDy = D_1 y$$

Similarly, for the second order derivative

$$\begin{aligned}\frac{d^2y}{dx^2} =& \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) \\ =& -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\end{aligned}$$

Thus, we have

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \Rightarrow \quad x^2D^2y = D_1(D_1 - 1)y.$$

Similarly, $x^3D^3y = D_1(D_1 - 1)(D_1 - 2)y$ and so on. In general, we have the relationship

$$x^nD^n = D_1(D_1 - 1)(D_1 - 2)\ldots(D_1 - n + 1)y$$

Substituting the above values of $x, xD, x^2D^2, \ldots, x^nD^n$ in the Equation (35.1), we get

$$[a_0D_1(D_1 - 1)\ldots(D_1 - n + 1) + \ldots + a_{n-2}D_1(D_1 - 1) + a_{n-1}D_1 + a_n]\, y = F\left(e^z\right) \quad (35.4)$$

The Equation (35.4) is a linear differential equation with constant coefficients which can solved with the methods discussed in previous lessons. Finally, by replacing z by $\ln x$ we obtain the desired solution of the given differential equation.

35.2 Example Problems

35.2.1 Problem 1

Solve the differential equation $(x^2D^2 + xD - 4)y = 0$.

Solution: Substituting $x = e^z \;\Rightarrow\; \ln x = z \;\Rightarrow\; xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$, the given equation reduces to

$$[D_1(D_1 - 1) + D_1 - 4]\, y = 0 \quad \Rightarrow \quad (D_1^2 - 4)y = 0$$

The roots of the corresponding characteristic equation are $m = 2, -2$. The required solution of the transformed equation is

$$y = c_1 e^{2z} + c_2 e^{-2z}$$

Putting $\log x = z$, we have the desired solution as

$$y = c_1 x^2 + c_2 x^{-2}.$$

Here c_1 and c_2 are arbitrary constants.

35.2.2 Problem 2

Find the general solution of the differential equation $(x^2 D^2 + y)y = 3x^2$.

Solution: Substituting $x = e^z$, the given equation reduces to

$$(D_1(D_1 - 1) + 1)y = 3e^{2z} \quad \Rightarrow \quad (D_1^2 - D_1 + 1)y = 3e^{2z}$$

The characteristic equation of this differential equation is

$$(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = (1 \pm i\sqrt{3})/2$$

The complimentary function is

$$\text{C.F.} = e^{z/2}\left[c_1 \cos\left(z\sqrt{3}/2\right) + \left(c_1 \sin z\sqrt{3}/2\right)\right]$$

Substituting $z = \ln x$, we get

$$\text{C.F.} = \sqrt{x}\left[c_1 \cos\left(\ln x\sqrt{3}/2\right) + c_1 \sin\left(\ln x\sqrt{3}/2\right)\right]$$

The particular integral of the transformed equation is

$$\text{P.I.} = \frac{1}{D_1^2 - D_1 + 1} 3e^{2z} = \frac{1}{2^2 - 2 + 1} 3e^{2z} = e^{2z}$$

Hence, the desired solution of the given differential equation is

$$y = \sqrt{x}\left[c_1 \cos\left(\ln x\sqrt{3}/2\right) + c_1 \sin\left(\ln x\sqrt{3}/2\right)\right] + x^2$$

35.3 Legendre's Homogeneous Linear Differential Equations

A linear differential equation of the form is

$$[(a+bx)^n a_0 D^n + a_1(a+bx)^{n-1}D^{n-1} + a_2(a+bx)^{n-2}D^{n-2} + ... + a_n]y = F(x), \quad (35.5)$$

where $a, b, a_1, a_2, ..., a_n$ are constants, and F is either a constant or a function of x only, is called a Legendre's homogeneous linear differential equation. Note that the index of $(a+bx)$ and the order of derivative is same in each term of such equation. To solve the Equation (35.5), we introduce a new independent variable z such that

$$a+bx = e^z, \quad \text{or} \quad \ln(a+bx) = z, \quad \text{so that} \quad b/(a+bx) = dz/dx. \quad (35.6)$$

Now, for the first order derivative we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{b}{(a+bx)}\frac{dy}{dz}$$

This implies

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} \quad \Leftrightarrow \quad (a+bx)Dy = bD_1 y$$

Similarly for the second order derivative we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{(a+bx)}\frac{dy}{dz}\right)$$

This can be further simplified to get

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\frac{d}{dx}\left(\frac{dy}{dz}\right)\\ &= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\end{aligned}$$

Substituting dz/dx from Equation (35.6), we obtain

$$\frac{d^2y}{dx^2} = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b^2}{(a+bx)^2}\frac{d^2y}{dz^2}$$

This gives us

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \quad \Leftrightarrow \quad (a+bx)^2D^2y = b^2D_1(D_1-1)y$$

In general, we have

$$(a+bx)^n D^n = b^n D_1(D_1-1)(D_1-2)\ldots(D_1-n+1)y$$

Substituting the above values of $(a+bx), (a+bx)D, (a+bx)^2D^2, \ldots, (a+bx)^nD^n$ in the Equation (35.5), we get the following linear differential equation with constant coefficients

$$\left[a_0 b^n D_1(D_1-1)...(D_1-n+1) + ... + a_{n-2}b^2 D_1(D_1-1) + a_{n-1}bD_1 + a_n\right] y = F\left(\frac{e^z-a}{b}\right)$$

The methods of solving this transformed equation are same as discussed in previous section.

35.3.1 Example

Solve the differential equation

$$(1+x)^4\frac{d^3y}{dx^3} + 2(1+x)^3\frac{d^2y}{dx^2} - (1+x)^2\frac{dy}{dx} + (1+x)y = \frac{1}{(1+x)}$$

Solution: Using $D = \dfrac{d}{dx}$ and dividing both sides by $(x+1)$, the given differential equation can be rewritten as

$$\left[(1+x)^3D^3 + 2(1+x)^2D^2 - (1+x)D + 1\right] y = (1+x)^{-2}.$$

This is the Legendre's homogeneous linear equation which can be solved by substituting

$$(1+x) = e^z \Leftrightarrow \ln(1+x) = z$$

This substitution readily implies

$$(1+x)D = D_1, \quad (1+x)^2D^2 = D_1(D_1-1), \quad (1+x)^3D^3 = D_1(D_1-1)(D_1-2)$$

The given differential equation reduces to

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) - D_1 + 1]\, y = e^{-2z}$$

or

$$\left(D_1^3 - D_1^2 - D_1 + 1\right) y = e^{-2z}$$

The characteristic equation of the corresponding homogeneous equation is

$$\left(m^3 - m^2 - m + 1\right) y = 0$$

The roots of the characteristics equations are $m = 1, 1, -1$. Hence the complimentary function of the transformed differential equation is

$$\text{C.F.} = (c_1 + c_2 z)e^z + c_3 e^{-z}$$

The particular integral of the transformed differential equation can be found as

$$\begin{aligned}
\text{P.I.} =& \frac{1}{(D_1^3 - D_1^2 - D_1 + 1)} e^{-2z} \\
=& \frac{1}{-2^3 - 2^2 + 2 + 1} e^{-2z} \\
=& -\frac{1}{9} e^{-2z}
\end{aligned}$$

Hence the general solution of the transformed differential equation is

$$y = (c_1 + c_2 z)e^z + c_3 e^{-z} - \frac{1}{9} e^{-2z}$$

Replacing z by $\ln(1 + x)$ we obtain the desired solution of the given differential equation

$$y = \left[c_1 + c_2 \ln(1 + x)\right](1 + x) + \frac{c_3}{(1 + x)} - \frac{1}{9} \frac{1}{(1 + x)}.$$

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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