Lesson 3

Indeterminate forms ; L'Hospital's Rule

3.1 Introduction

Consider the following limits $\lim_{x\to 4}\frac{x^2-16}{x-4}$ and $\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$

In the first limit if we put $x=4$ we will get $\frac{0}{0}$ and in the second limit if we "plugged" in infinity we get $\frac{\infty}{-\infty}$ (recall that as x goes to infinity a polynomial will behave in the same fashion that it's largest power behaves). Both of these are called Indeterminate form.

3.1.1 Indeterminate forms

First limit can be found by the factorizing the numerator cancelling the common factor. That is

$$\lim_{x\to 4}\frac{x^2-16}{x-4}$$

$$=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}$$

$$=\lim_{x\to 4}(x+4)$$

$$=8$$

The second limit can be evaluated as:

$$\lim_{x\to\infty} \frac{4x^2-5x}{1-3x^2}$$

$$= \lim_{x\to\infty} \frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}$$

$$= -\frac{4}{3}$$

However what about the following two limits. $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x\to\infty} \frac{e^x}{x^2}$, This first is a $\frac{0}{0}$ indeterminate form, but we can't factor this one. The second is an $\frac{\infty}{\infty}$ indeterminate form, but we can't just factor an x^2 out of the numerator. Does there exists some method to evaluate the limits? The answer is yes. By (L'Hospital's Rule).

Suppose that we have one of the following cases,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Theorem 3.1: Suppose the functions $f(x)$ and $g(x)$ in $[a,b]$, satisfy the Cauchy Theorem and $f(a) = g(a) = 0$, then if the ratio $\frac{f'(x)}{g'(x)}$ has a limit as $x \to a$, there also exists $\lim_{x\to a} \frac{f(x)}{g(x)}$, and $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = A$.

Proof.: On the interval $[a,b]$ take some point $x \neq a$. Applying the Cauchy's mean value theorem we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

where ξ is a number lies between a and x. But it is given that $f(a) = g(a) = 0$ and so

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \quad \text{..................(1)}$$

If $x \to a$, then $\xi \to a$, since ξ lies between x and a. Suppose if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, by (1) $\lim_{\xi\to a} \frac{f'(\xi)}{g'(\xi)}$ exists and is equal to A. Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)}$$

$$= \lim_{\xi \to a} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = A$$

and, finally,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note 3.1: The theorem also holds for the case where the functions $f(x)$ and $g(x)$ are not defined at $x = a$, but $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. We can make them to be continuous at $x = a$ by redefine $f(a) = \lim_{x \to a} f(x) = 0$, $g(a) = \lim_{x \to a} g(x) = 0$, since $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not depend on whether the function $f(x)$ and $g(x)$ are defined at $x = a$.

Note 3.2: If $f'(a) = g'(a) = 0$ and the derivatives $f'(x)$ and $g'(x)$ satisfy the conditions that we imposed by the theorem on the functions $f(x)$ and $g(x)$, then applying the L'Hospital rule $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$, and so forth.

Note 3.3: If $g'(x) = 0$, but $f'(x) \neq 0$, then the theorem is applicable to the reciprocal ratio $\frac{g(x)}{f(x)}$, which tends to zero as $x \to a$. Hence, the ratio $\frac{f(x)}{g(x)}$ tends to infinity.

Example 3.1: $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= \frac{2}{1} = 2.$$

Note 3.4: The L'Hospital rule is also applicable if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

Put $x = \frac{1}{z}$, we see that $z \to 0$ as $x \to \infty$ and therefore $\lim_{z\to 0} f\left(\frac{1}{z}\right) = 0$, and $\lim_{z\to 0} g\left(\frac{1}{z}\right) = 0$. Applying the L'Hospital rule to the ratio

$\frac{f\left(\frac{1}{z}\right)}{g\left(\frac{1}{z}\right)}$, we find that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{z\to 0}\frac{f(\frac{1}{z})}{g(\frac{1}{z})}$$

$$= \lim_{z\to 0}\frac{f^{'}(\frac{1}{z})(-\frac{1}{z^2})}{g^{'}(\frac{1}{z})(-\frac{1}{z^2})}$$

$$= \lim_{z\to 0}\frac{f^{'}(\frac{1}{z})}{g^{'}(\frac{1}{z})} = \lim_{x\to\infty}\frac{f^{'}(x)}{g^{'}(x)}$$

which proves the results.

We also stated in earlier that if both $f(x)$ and $g(x)$ approaching infinity as $x \to a$ (or $x \to \infty$), the L'Hospital rule is also applied.

Example 3.2: Find $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}$ $(\frac{\infty}{\infty})$

Solution:

Taking derivative both numerator and denominator five times we obtain: **Ans: 3**

Other Indeterminate forms :

The other indeterminate forms reduce to the following cases. (a) $0.\infty$ (b) 0^0 (c) ∞^0 (d) 1^∞ (e)$\infty - \infty$.

(a) Let $\lim_{x\to a} f(x) = 0, \lim_{x\to a} g(x) = \infty$, it is required to find

$\lim_{x\to a} [f(x)g(x)]$,

i.e. the indeterminate form $0.\infty$. Now

$$\lim_{x\to a} [f(x)g(x)]$$

$$= \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$

or $f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}}$ If $\lim f(x) = \infty, x \to a$ & $\lim g(x) = 0, x \to a$

which is $(\frac{0}{0})$- form or one can write

$$\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$$

$(\frac{\infty}{\infty})$- form

Example 3.3

$$\lim_{x\to 0} x^n \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x^n}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}} = \lim_{x\to 0} \frac{x^n}{n} = 0$$

b) Let $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, it is required to find $\lim_{x\to a}[f(x)]^{g(x)}$. Put $y = [f(x)]^{g(x)}$. Taking logarithms of both sides of it, we have

$$\ln y = g(x)[\ln f(x)]$$

$$\lim_{x\to a} \ln y = \ln \lim_{x\to a} y$$

(by the continuity of $\ln y$) and if $\ln \lim_{x\to a} y = b \Rightarrow e^b = \lim_{x\to a} y$.

Similarly we can find the Indeterminate form $\infty^0, 1^\infty$

Example 3.4: $\lim_{x\to 0} x^x$ Solution: Put $y = x^x$,

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} x \ln x$$

$$= \lim_{x\to 0} (x \ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$$

$$= 0$$

So $\lim_{x\to 0} y = e^0 = 1$.

Example 3.5: Find the $\lim_{x\to 0} (\frac{1}{x})^{\tan x}$

Ans: 1

Example 3.6 Using Taylor's formula compute

$$\lim_{x\to 0}\frac{x-\sin x}{e^{x}-1-x-\frac{x^{2}}{2}}$$

Ans: 1

Questions: Answer the following questions.

Evaluate the following limits :

1. $\lim_{x\to 1}\frac{x-1}{x^{n}-1}$

2. $\lim_{x\to 0}\frac{e^{x^{2}}-1}{\cos x-1}$

3. $\lim_{x\to 0}\frac{\sin x}{\sqrt{1-\cos x}}$

4. $\lim_{x\to 0}\frac{e^{y}+\sin y-1}{\ln(1+y)}$

5. $\lim_{x\to 1}\frac{\ln(x-1)-x}{\tan\frac{\pi}{2x}}$

6. $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln x}\right]$

7. $\lim_{x\to 0}(\cot x)^{\frac{1}{\ln x}}$

8. $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$

Ans.: 1. 1, 2. -2, 3. Limit does not exist, 4. 2, 5. 0, 6. $\frac{1}{2}$, 7. $\frac{1}{6}$ & 8. 1

Keywords: Indeterminate forms ; L'Hospital's Rule.

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Suggested Readings

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