Lesson 29

Exact Differential Equations: Integrating Factors

In general, equations of the type $M(x,y)dx + N(x,y)dy = 0$ are not exact. However, it is sometimes possible to transform the equation into an exact differential equation multiplying it by a suitable function $I(x,y)$. That is, if $I(x,y)$ is an integrating factor then the differential equation

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

becomes exact. A solution to the above equation is obtained by solving the exact differential equation as in the previous lesson. Note that the given equation may have several integrating factors. This is exactly the procedure we have used for solving linear differential equations in earlier lesson. Here we deal with more general differential equation.

29.1 Rule I: By Inspection

There is not much theory behind finding integrating factor by inspection. This method works based on recognition of some standard exact differentials that occur frequently in practice. The following list of exact differentials would be quite useful in solving exact differential equations:

$$(i) \quad d(xy) = ydx + xdy$$

$$(ii) \quad d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \quad \text{or} \quad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(iii) \quad d\left(\ln\frac{y}{x}\right) = \frac{xdy - ydx}{xy} \quad \text{or} \quad d\left(\ln\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$$

$$(iv) \quad d\left(\arctan\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2} \quad \text{or} \quad d\left(\arctan\frac{x}{y}\right) = \frac{ydx - xdy}{y^2 + x^2}$$

$$(v) \quad d(\ln xy) = \frac{ydx + xdy}{xy}$$

29.1.1 Example

Solve the differential equation $y(y^2+1)dx + x(y^2-1)dy$.

Solution: The given equation can be rewritten as

$$y^2(ydx + xdy) + ydx - xdy$$

This is further rewritten as

$$(ydx + xdy) + \left(\frac{ydx - xdy}{y^2}\right) = 0$$

Using standard differential forms given above we get

$$d(xy) + d\left(\frac{x}{y}\right) = 0$$

Integrating the above equation, the desired solution is given as

$$xy^2 + x = cy$$

Here c is an arbitrary constant.

29.2 Rule II: $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$

If the equation $Mdx+Ndy = 0$ is homogeneous and $Mx+Ny \neq 0$, then $I(x, y) = \dfrac{1}{(Mx + Ny)}$ is an integrating factor. In order to prove the result, we need to show that

$$\frac{Mdx + Ndy}{Mx + Ny} = d\,(\text{some function } x \text{ and } y)$$

Rewriting $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Multiplying by proposed integrating factor we get

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{(Mx - Ny)}{(Mx + Ny)}\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\} \tag{29.1}$$

Given that $M(x, y)$ and $N(x, y)$ are homogeneous functions of some degree n, i.e., $M(tx, ty) = t^n M(x, y)$ and $N(x, y) = t^n N(x, y)$. Then

$$M\left(\frac{x}{y}, 1\right) = M\left(\frac{1}{y}x, \frac{1}{y}y\right) = \frac{1}{y^n}M(x, y) \quad \Rightarrow \quad M(x, y) = y^n M\left(\frac{x}{y}, 1\right)$$

Similarly, we get

$$N(x,y) = y^n N\left(\frac{x}{y}, 1\right)$$

Now consider

$$\frac{(Mx - Ny)}{(Mx + Ny)} = \frac{y^n x M\left(\frac{x}{y},1\right) - y^n y N\left(\frac{x}{y},1\right)}{y^n x M\left(\frac{x}{y},1\right) + y^n y N\left(\frac{x}{y},1\right)} = \frac{\frac{x}{y} M\left(\frac{x}{y},1\right) - N\left(\frac{x}{y},1\right)}{\frac{x}{y} M\left(\frac{x}{y},1\right) + N\left(\frac{x}{y},1\right)} = f\left(\frac{x}{y}\right)$$

Going back to the Equation (29.1), we have

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{d\left(\ln(xy)\right) + f\left(\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right)\right\}$$

Rewriting $f\left(x/y\right) = f\left(\exp(\ln(x/y))\right)$ and defining $g(x) := f(\exp(x))$, the above equation becomes

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{d\left(\ln(xy)\right) + g\left(\ln(x/y)\right) d\left(\ln\frac{x}{y}\right)\right\}$$

Hence, we have shown that

$$\frac{Mdx + Ndy}{Mx + Ny} = d\left[\frac{1}{2}\ln(xy) + \frac{1}{2}\int g\left(\ln\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right)\right]$$

Thus $\dfrac{1}{Mx + Ny}$ is an integrating factor of the homogenous differential equation $Mdx + Ndy = 0$.

29.2.1 Example

Solve the differential equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: The given equation is a homogeneous differential equation. Comparing it with $Mdx + Ndy = 0$, we have $M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$. Since

$$Mx + Ny = (x^2y - 2xy^2)x - y(x^3 - 3x^2y) = x^2y^2 \neq 0,$$

the integrating factor is

$$\frac{1}{(Mx + Ny)} = \frac{1}{x^2y^2}$$

Multiply by the integrating factor, the given differential equation becomes

$$(1/y - 2/x)dx - (x/y^2 - 3/y)dy = 0$$

This is now exact and can be rewritten as

$$\frac{ydx - xdy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0 \quad \Rightarrow \quad d\left(\frac{x}{y}\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

Integrating the above equation we obtain the desired solution as

$$x - 2y\ln x + 3y\ln y = cy$$

29.3 Rule III: $Mdx + Ndy = 0$ is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$

If the equation $Mdx+Ndy = 0$ is of the form $f_1(xy)ydx+f_2(xy)xdy = 0$, then $\frac{1}{(Mx - Ny)}$ is an integrating factor provided $Mx - Ny \neq 0$. Similar to rule II we now show that

$$\frac{Mdx + Ndy}{Mx - Ny} = d\,(\text{some function } x \text{ and } y)$$

Again, rewriting $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Now dividing by $Mx - Ny$ we get

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{\frac{(Mx + Ny)}{Mx - Ny}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Using $M = f_1(xy)y$ and $N = f_2(xy)x$ we obtain

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{\frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\}$$

Let $f(xy) := \frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}$ and $g(x) := f(\exp(x))$, the above equation reduces to

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{f(xy)d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\} = \frac{1}{2}\left\{g(\ln xy)d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\}$$

This shows that

$$\frac{Mdx + Ndy}{Mx - Ny} = d\left[\frac{1}{2}\int g(\ln xy)d\,(\ln xy) + \frac{1}{2}\left(\ln\frac{x}{y}\right)\right]$$

29.3.1 Example

Solve $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$.

Solution: Comparing with $Mdx+Ndy = 0$, we have $M = y(x^2y^2+2)$ and $N = x(2-2x^2y^2)$. The given equation is of the form

$$f_1(xy)ydx + f_2(xy)xdy = 0$$

and we have

$$Mx - Ny = xy(x^2y^2 + 2) - xy(2 - 2x^2y^2) = 3x^3y^3 \neq 0$$

Therefore, multiplying the equation by $1/3x^3y^3$, we obtain

$$(1/3x + 2/(3x^3y^2))dx + (2/(3x^2y^3) - 2/3y)dy = 0$$

This is an exact differential equation which can be solved with the technique discussed in previous lesson.

29.4 Rule IV: Most general approach

Now we discuss the most general approach of finding integrating function. The idea is to multiply the given differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{29.2}$$

by a function $I(x, y)$ and then try to choose $I(x, y)$ so that the resulting equation

$$I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0 \tag{29.3}$$

becomes exact. The above equation is exact if and only if

$$\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x} \tag{29.4}$$

If a function $I(x, y)$ satisfying the partial differential Equation (29.4) can be found, then (29.3) will be exact. Unfortunately, solving Equation (29.4), is as difficult to solve as the original Equation (29.2) by some other methods. Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The cases we will consider are: (i) an integrating factor I that is either as function of x only, or (ii) a function of y only.

Let us determine necessary conditions on M and N so that (29.2) has an integrating factor I that depends on x only. Assuming that I is a function of x only, then Equation (29.4) reduces to

$$IM_y = IN_x + N\frac{dI}{dx} \quad \Rightarrow \quad \frac{dI}{dx} = \frac{IM_y - IN_x}{N} \tag{29.5}$$

If $(M_y - N_x)/N$ is a function of x only, say f(x), then there is an integrating factor I that also depends only on x which can be found by solving (29.5) as $I(x) = e^{\int f(x)dx}$. A similar procedure can be used to determine a condition under which Equation (29.2) has an integrating factor depending only on y. To conclude, we have:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$, then $I(x) = e^{\int f(x)dx}$ is an I.F.

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $I(y) = e^{\int f(y)dy}$ is an I.F.

29.5 Example Problems

29.5.1 Problem 1

Find an integrating factor of $(x^2 + y^2 + x)dx + xydy = 0$ **Solution:** Comparing with $Mdx + Ndy = 0$, we have

$$M = (x^2 + y^2 + x) \text{ and } N = xy$$

Further, note that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x}$$

is a function of x alone. Hence, the integrating factor of the given problem is $e^{\int 1/xdx} = x$.

29.5.2 Problem 2

Find an integrating factor of $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Solution: Compare with $Mdx + Ndy = 0$, we get

$$M = (2xy^4e^y + 2xy^3 + y) \text{ and } N = (x^2y^4e^y - x^2y^2 - 3x)$$

Also, note that

$$\frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right) = -\frac{4}{y}$$

is a function of y alone. Hence the integrating factor of the given problem is $e^{\int -4/y dy} = 1/y^4$.

Suggested Readings

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