Lesson 28

Exact Differential Equation of First Order

This lesson provides an overview of exact differential equation. A necessary condition for a differential equation to be exact will be derived. Then different solution techniques will be discussed. Several examples to clarify the ideas will be supplemented.

28.1 Exact Differential Equation of First Order

If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x, y)$ such that

$$d(f(x, y)) = Mdx + Ndy,$$

or equivalently

$$\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = Mdx + Ndy.$$

28.1.1 Theorem

The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \tag{28.1}$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{28.2}$$

Proof: First we proof that the condition (28.2) is necessary. To prove we let the Equation (28.1) to be exact. Then, by definition, there exists $f(x, y)$ such that

$$\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = Mdx + Ndy. \tag{28.3}$$

Equating coefficients of dx and dy in Equation (28.3), we get

$$M = \frac{\partial f}{\partial y}, \tag{28.4}$$

$$N = \frac{\partial f}{\partial x}. \tag{28.5}$$

To eliminate the unknown $f(x, y)$ from above equations, we assume that the 2nd order partial derivatives of f are continuous. We now differentiate (28.4) and (28.5) w.r.t. x and y respectively as

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

This implies

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, if (28.1) is exact, M and N satisfy (28.2).

Now we show that the condition is sufficient. Suppose (28.2) holds and show that (28.1) is exact. For this we find a function $f(x, y)$ such that

$$d(f(x, y)) = M dx + N dy.$$

Let $g(x, y) = \int M dx$ be the partial integral of M such that $\frac{\partial g}{\partial x} = M$. We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is function of y only. This is clear because

$$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$$

Assuming $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$ and using Equation (28.2) we get

$$\begin{aligned}\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) =& \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \\ =& \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}(\frac{\partial g}{\partial x}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.\end{aligned}$$

Take, $f(x, y) = g(x, y) + \int (N - \frac{\partial g}{\partial y}) dy$. Hence taking total differentiation of this equation gives

$$\begin{aligned} df = dg + (N - \frac{\partial g}{\partial y}) dy &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + N dy - \frac{\partial g}{\partial y} dy, \\ &= (\frac{\partial g}{\partial x}) dx + N dy = M dx + N dy, \end{aligned}$$

Thus, if Equation (28.2) is satisfied, Equation (28.1) is an exact equation. ■

28.2 Example Problems

28.2.1 Problem 1

Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = (x^2 - 4xy - 2y^2), \qquad N = (y^2 - 4xy - 2x^2)$$

Therefore

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}$$

Hence, the given equation is exact and hence there exists a function $f(x, y)$ such that

$$d(f(x, y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

which implies

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)$$

Integration of the first of above equations with respect to x gives

$$f = \frac{1}{3}x^3 - 2x^2y - 2y^2x + c_1(y)$$

where $c_1(y)$ is an arbitrary function of y only. Differentiating the above f with respect to y and using $\frac{\partial f}{\partial y} = N(x, y)$ we get

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = +y^2 - 4xy - 2x^2$$

This implies

$$c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence the solution is given by

$$f(x, y) = c_3 \Rightarrow x^3 - 6xy(x + y) + y^3 = c$$

Here c_2, c_3 and c are constants of integration.

28.2.2 Problem 2

Determine whether the differential equation $(x + \sin y)dx + (x \cos y - 2y)dy = 0$ *is exact and solve it.*

Solution: For given equation we have

$$M(x, y) = (x + \sin y) \quad \text{and} \quad N(x, y) = (x \cos y - 2y) \tag{28.6}$$

Now we check

$$\frac{\partial M}{\partial y} = \cos y = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact. For the solution we seek a function $f(x, y)$ so that

$$\frac{\partial f}{\partial x} = (x + \sin y) \quad \text{and} \quad \frac{\partial f}{\partial y} = (x \cos y - 2y)$$

From the first relation we get

$$f(x, y) = \frac{x^2}{2} + x \sin y + c_1(y)$$

Differentiating w.r.t. y and using the second relation of (28.6) we get

$$x \cos y + c_1'(y) = x \cos y - 2y \quad \Rightarrow \quad c_1'(y) = -2y \Rightarrow c_1(y) = -y^2 + c_2$$

Therefore, we have

$$f(x, y) = \frac{x^2}{2} + x \sin y - y^2 + c_2$$

Then the solution of the given differential equation

$$f(x, y) = c_3 \quad \Rightarrow \quad \frac{x^2}{2} + x \sin y - y^2 = c.$$

28.2.3 Problem 3

Solve the differential equation $(2y^2x - 2y^3)dx + (4y^3 - 6y^2x + 2yx^2)dy$

Solution: First we check the exactness of the equation by

$$\frac{\partial M}{\partial y} = 4xy - 6y^2 = \frac{\partial N}{\partial x}$$

So the equation is exact. Then, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = (2y^2x - 2y^3) \quad \text{and} \quad \frac{\partial f}{\partial y} = (4y^3 - 6y^2x + 2yx^2)$$

This gives

$$f(x, y) = (y^2x^2 - 2xy^3) + c_1(y) \quad \Rightarrow \quad \frac{\partial f}{\partial y} = (2yx^2 - 6xy^2) + c_1'(y)$$

This implies

$$c_1'(y) = 4y^3 \quad \Rightarrow \quad c_1(y) = y^4 + c_2$$

Hence the solution is

$$f(x, y) = c_3 \quad \Rightarrow \quad y^2x^2 - 2xy^3 + y^4 = c.$$

28.2.4 Problem 4

Solve that the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0.$ *is not exact and hence it cannot be solve by the method discussed above.*

Solution: For the given differential equation we have

$$\frac{\partial M}{\partial y} = 3x + 2y, \quad and \quad \frac{\partial N}{\partial x} = 2x + y;$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is not exact.

Now we see that it cannot be solved by the procedure described previously where we seek a function f such that

$$\frac{\partial f}{\partial x} = 3xy + y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + xy \tag{28.7}$$

Integration of the first relation gives

$$f(x, y) = \frac{3}{2}x^2y + xy^2 + c_1(y)$$

where $c_1(y)$ is an arbitrary function of y only. Now we differentiate the above equation with respect to y and set the resulting expression equals to $x^2 + xy$ from the second relation of (28.7) as

$$\frac{3}{2}x^2 + 2xy + c_1'(y) = x^2 + xy$$

This provides

$$c_1'(y) = -\frac{1}{2}x^2 - xy$$

Since the right side of the above depends on x as well as on y, it is impossible to solve this equation for $c_1(y)$. Thus there is no $f(x,y)$ exists and hence the given differential equation cannot be solved in this way.

Suggested Readings

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