Lesson 26

Differential Equation of First Order

In this lesson we present solution techniques of differential equations of first order and first degree. We shall mainly discuss differential equation of variable separable form, homogeneous equations and equations reducible to homogeneous form.

There are two standard forms of differential equations of first order and first degree, namely,

$$\frac{dy}{dx} = f(x, y) \qquad \text{or} \qquad Mdx + Ndy = 0$$

Here M and N are functions of x and y, or constants. We discuss here some special forms of these equations where exact solution can easily be obtained.

26.1 Separation of Variables

If in a differential equation, it is possible to get all the functions x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable. In other words if a differential equation can be written in the form $F(x)dx + G(y)dy = 0$, we say variables are separable and its solution is obtained by integrating the equation as

$$\int F(x)dx + \int G(y)dy = c,$$

where c is a integration constant.

26.2 Example Problems

26.2.1 Problem 1

Solve $\dfrac{dy}{dx} = e^{x+y} + x^2 e^y$.

Solution: For separating variables, we rewrite the given equation as

$$e^{-y}dy = (e^x + x^2)dx.$$

Integrating the above equation we have

$$-e^{-y} = e^x + x^3/3 + c,$$

where c is an arbitrary constant.

26.2.2 Problem 2

Solve $3e^x \tan y dx + (1 - e^x)\sec^2 y dy = 0$.

Solution: Separating the variables, we get

$$\frac{3e^x}{1 - e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0.$$

Integration gives

$$-3\log(1 - e^x) + \log(\tan y) = \log c,$$

where c is an arbitrary constant.

26.3 Equations Reducible to Separable Form

Differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c) \quad or \quad \frac{dy}{dx} = f(ax + by)$$

can be reduced by the substitution $ax + by + c = v$ or $ax + by = v$ to an equation in which variables can be separated.

26.3.1 Example

Solve $\frac{dy}{dx} = \sec(x + y)$.

Solution: Let, $x + y = v$ so that

$$\frac{dy}{dx} = \frac{dv}{dx} - 1. \tag{26.1}$$

Using (26.1), the given differential equation becomes

$$\frac{dv}{dx} = \sec v + 1. \tag{26.2}$$

This equation is of separable form. Thus we have

$$dx = \frac{1}{\sec v + 1}dv \quad \Rightarrow \quad dx = \frac{2\cos^2(v/2) - 1}{1 + 2\cos^2(v/2) - 1}dv$$

Further simplifications gives

$$dx = \left(1 - \frac{1}{2}\sec^2(v/2)\right) dv$$

Integrating and substituting the value of v, we obtain $y - \tan\frac{1}{2}(x+y) = c$.

26.4 Homogeneous Differential Equation

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form

$$\frac{dy}{dx} = f(y/x).$$

These equations can be solved by letting $y/x = v$ and differentiating with respect to x as

$$v + x\frac{dv}{dx} = f(v) \quad \Rightarrow \quad x\frac{dv}{dx} = f(v) - v.$$

Then, separating variables, we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

Integrating the above equation we obtain

$$\log x + c = \int \frac{dv}{f(v) - v},$$

where c is an arbitrary constant. The solution is obtained by replacing variable v by y/x.

26.4.1 Example

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

Solution: Since the right hand side of the given equation is function of y/x alone, the given problem is homogeneous equation. Substituting $y/x = v$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{26.3}$$

the given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v \quad \rightarrow \quad \frac{dx}{x} = \frac{\cos v}{\sin v} dv$$

Integrating and substituting the value of v, we get the solution as

$$cx = \sin\left(\frac{y}{x}\right),$$

where c is an arbitrary constant.

26.5 Equations Reducible to Homogeneous Form

Equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}, \qquad \frac{a}{a'} \neq \frac{b}{b'} \tag{26.4}$$

can be reduced to homogeneous form. The procedure is as follows:

Take

$$x = X + h \text{ and } y = Y + k$$

where X, Y are new variables and h, k are constants to be chosen so that the resulting equation in X, Y becomes homogeneous. From above we have $dx = dX$, and $dy = dY$, so that $dy/dx = dY/dX$. Now the given differential equation in new variables becomes

$$\frac{dX}{dY} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \tag{26.5}$$

In order to make (26.5) homogeneous, the constant h and k must satisfy the following algebraic equations

$$ah + bk + c = 0 \quad , \quad a'h + b'k + c' = 0 \tag{26.6}$$

Solving equations (26.6), we obtain

$$h = \frac{bc' - b'c}{ab' - a'b} \quad , \quad k = \frac{ca' - c'a}{ab' - a'b} \tag{26.7}$$

provided $ab' - a'b \neq 0$. Knowing h and k we have

$$X = x - h, \qquad Y = y - k. \tag{26.8}$$

The Equation (26.5) now reduces to

$$\frac{dY}{dX} = \frac{aX + b(Y/X)}{a' + b'(Y/X)} \tag{26.9}$$

which is a homogeneous equation in X and Y which can be solved by substituting $Y/X = v$. After getting solution in X and y, we remove X and Y using (26.8) and obtain solution in terms of x and y.

26.5.1 Example

Solve the differential equation

$$\frac{dy}{dx} = \frac{(x + y + 4)}{(x - y - 6)}$$

Solution: Let $x = X + h, \quad y = Y + k,$ so that $dy/dx = dY/dX$ and using this, the given differential equation reduces to

$$\frac{dy}{dx} = \frac{X + Y + (h + k + 4)}{X - Y + (h - k - 6)}. \tag{26.10}$$

Choose h and k such that $h + k + 4 = 0, \quad h - k - 6 = 0$, and by solving, we get $h = 1$ and $k = -5$. New variables becomes $X = x - 1$ and $Y = y + 5$. Using this into (26.10), we obtain

$$\frac{dY}{dX} = \frac{1 + Y/X}{1 + Y/X}. \tag{26.11}$$

Substituting

$$Y = Xv \text{ and } \frac{dY}{dX} = v + X\frac{dv}{dX}$$

the Equation (26.11) becomes

$$\frac{dX}{X} = \frac{1 - v}{1 + v^2}dv = \frac{dv}{1 + v^2}dv - \frac{vdv}{1 + v^2}. \tag{26.12}$$

Integrating the above equation, we get

$$\log X = \tan^{-1} v - (1/2)\log(1 + v^2) + (1/2)\log c$$

Further simplifications gives

$$2\log X + \log(1 + Y^2/X^2) - \log c = 2\tan^{-1}(Y/X), \quad \text{as} \ \ v = Y/X$$

Thus, we get

$$X^2 + Y^2 = ce^{2\tan^{-1}(Y/X)};$$

Replacing X and Y as $X = x - 1$ and $Y = y + 5$ we obtain the general solution as

$$(x-1)^2 + (y+5)^2 = ce^{2\tan^{-1}((y+5)/(x-1))}.$$

Suggested Readings

Dubey, R. (2010). Mathematics for Engineers (Volume II). Narosa Publishing House. New Delhi.

Weir, M.D., Hass, J. and Giordano, F.R. (2005). Thomas Calculus. Eleventh Edition. Pearson Education. New Delhi.

McQuarrie, D.A. (2009). Mathematical Methods for Scientist and Engineers. First Indian Edition. Viva Books Pvt. Ltd. New Delhi.

Raisinghania, M.D. (2005). Ordinary & Partial Differential Equation. Eighth Edition. S. Chand & Company Ltd., New Delhi.

Kreyszig, E. (1993). Advanced Engineering Mathematics. Seventh Edition, John Willey & Sons, Inc., New York.

Arfken, G.B. (2001). Mathematical Methods for Physicists. Fifth Edition, Harcourt Academic Press, San Diego.

Grewal, B.S. (2007). Higher Engineering Mathematics. Fourteenth Edition. Khanna Publishilers, New Delhi.

Edwards, C.H., Penney, D.E. (2007). Elementary Differential Equations with Boundary Value Problems. Sixth Edition. Pearson Higher Ed, USA.

Piskunov, N. (1996). Differential and Integral Calculus (Volume - 2). First Edition. CBS Publisher, Moscow.