Lesson 25

Introduction

In this lesson we introduce basic concepts of theory of ordinary differential equations. Formation of the differential equation from a given family of curves is explained. Different types of solutions are defined. The given definitions are supplemented by some simple examples.

25.1 Differential Equations

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation. An ordinary differential equation of order n is defined by the relation

$$F(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 0 \tag{25.1}$$

where $x^{(n)}$ stands for the nth derivative of unknown function $x(t)$ with respect to the independent variable t. For example

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \tag{25.2}$$

$$\frac{dx}{dt} = x + \sin x. \tag{25.3}$$

25.1.1 Order of a Differential Equation

The order of a differential equation is referred to the highest order derivative involved in the differential equation. For example, the order of the differential Equation (25.2) is four.

25.1.2 Degree of Differential Equation

The degree of a differential equation is the degree of the highest order derivative which occurs in it; after the differential equation has been made free from radicals and fractions as far as derivatives are concerned, e.g. in differential Equation (25.2), the degree is one.

25.1.3 Linear and Nonlinear Differential Equation

A differential equation is called linear if (a) every dependent variable and every derivative involved occurs in first degree only, and (b) no product of dependent variables and/or derivatives occur. A differential is not linear is called nonlinear. For examples, Equation (25.2) is linear and (25.3) is nonlinear.

25.2 Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution of differential equation. For example, $y = e^{2x}$ is a solution of $y' = 2y$.

25.2.1 Example

Show that $y = A/x + B$ *is solution of*

$$y'' + \left(\frac{2}{x}\right) y' = 0$$

Solution: We have the differential equation

$$y'' + \left(\frac{2}{x}\right) y' = 0. \tag{25.4}$$

Also given that

$$y = A/x + B. \tag{25.5}$$

Differentiating (25.5) w.r.t. x

$$y' = -A/x^2. \tag{25.6}$$

Differentiating (25.6) w.r.t. x

$$y'' = 2A/x^3. \tag{25.7}$$

Substituting (25.6) and (25.7) into (25.4), we have

$$\frac{2A}{x^3} - \frac{2A}{x^3} = 0.$$

25.2.2 Complete, Particular and Singular Solutions

Let

$$F(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 0 \tag{25.8}$$

be an n-th odder differential equation.

- A solution of (25.8) containing n independent constants is called *general solution*.
- A solution of (25.8) obtained from a general solution by giving particular value to one or more of the n independent arbitrary constants is called *particular solution*.
- A solution which cannot be obtained from any general solution by any choice of the n independent arbitrary constants is called *singular solution*.

25.3 Formation of Differential Equations

An n-parameter family of curves is a set of relations of the form $\{(x, y) : f(x, y, c_1, c_2, ..., c_n) = 0\}$, where f is real valued function of $x, y, c_1, c_2, ..., c_n$ and each c_i $(i = 1, 2, ...n)$ ranges over an interval of real values.

Suppose we are given a family of curves containing n arbitrary constants. Then by differentiating it successively n times and eliminating all arbitrary constants from the $(n + 1)$ equations we obtain an nth order differential equation whose solution is the given family of curves. We now illustrate the procedure of forming differential equations with the help of some examples.

25.4 Example Problems

25.4.1 Problem 1

Find the differential equation of the family of curves $y = e^{mx}$, where m is an arbitrary constant.

Solution: We have the family of curves

$$y = e^{mx}. \tag{25.9}$$

Differentiating (25.9) w.r.t x, we get

$$y' = me^{mx}. \tag{25.10}$$

Now, we eliminate m from (25.9) and (25.10) and using $m = \log_e y$, we obtain the required differential equation as

$$y' = y \log_e y.$$

25.4.2 Problem 2

Obtain the differential equation satisfied by the family of circles $x^2 + y^2 = a^2$, where a is an arbitrary constant.

Solution: The family of circles is given as

$$x^2 + y^2 = a^2. \tag{25.11}$$

Differentiating (25.11) w.r.t x, we get

$$x + yy' = 0,$$

which is the required differential equation.

25.4.3 Problem 3

Obtain the differential equation satisfied by $xy = ae^x + be^{-x} + x^2$, where a and b are an arbitrary constant.

Solution: Given family of curves

$$xy = ae^x + be^{-x} + x^2. \tag{25.12}$$

Differentiating (25.12) w.r.t x, we get

$$xy' + y = ae^x - be^{-x} + 2x, \tag{25.13}$$

Differentiating (25.14) w.r.t x and using (25.14), we get

$$xy'' + 2y' = (xy - x^2) + 2, \tag{25.14}$$

which is the required differential equation.

Remark: *From the above examples we observed that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation and is the same as its order. Hence a general solution of an nth order differential equation will contain n arbitrary constant.*

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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