Lesson 24

The Beta Function

24.1 Introduction

In this Lesson we shall introduce a useful function of two variables known as beta function. Its usefulness is considerably overshadowed by that of gamma function. In fact, we shall show that it can be evaluated in terms of the latter function. As consequence, it would be unnecessary to introduce it as a new function. Since it occurs so frequently in analysis, a special designation for it is accepted.

Definition 2.2

For *x*, *y* positive we define the Beta function by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Using the substitution $u = 1 - t$ it is easy to see that

Theorem 24.1. $B(x,y) = B(y,x).$

Here we say the beta function is symmetric.

To evaluate the Beta function we usually use the Gamma function. To find their relationship, one has to do a rather complicated calculation involving change of variables (from rectangular into tricky polar) in a double integral.

When x and y are positive integers, it follows from the definition of the gamma function Γ that:

$$\mathrm{B}(x,y) = \frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

Theorem 24.2. For $0 < x < \infty, 0 < y < \infty$,

$$\beta(x,y) = \int_{0+}^{\infty} (\sin t)^{2x-1} (\cos t)^{2y-1} dt$$

To prove this set $t = \sin^2 u$ in the integral.

$$\mathrm{B}(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\,d\theta, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

Theorem 24.3. For $0 < x < \infty, 0 < y < \infty$,

$$\beta(x,y) = \int_{0+}^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt$$

Here the change of variable $t = u(1+u)^{-1}$ suffices.

It has many other forms, including:

Theorem. For $0 < x < \infty, 0 < y < \infty$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

Proof : When x and y are arbitrary positive numbers, the proof proceeds as follows. From the double integral of the nonnegative function $t^{2x-1}u^{2y-1}e^{-t^2-u^2}$ over the three regions D_1, D_2 and S of figure 1 of Lesson 23. Now, however, t and u are the variables, however, t and u are the variables x and y positive constants. We have relation (23.5) of Lesson 23 as before. Again we evaluate the central double integral by iteration in rectangular coordinates: the other two, in polar coordinates:

$$\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \int_0^R e^{-r^2} r^{2x+2y-1} dr < \int_0^R t^{2x-1} e^{-t^2} dt \int_0^R u^{2y-1} e^{-u^2} du$$

$$< \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \int_0^{R\sqrt{2}} e^{-r^2} r^{2x+2y-1} dr$$

Now, if we let R become infinite and use Theorems 23.4 and 24.3, we obtain

$$\frac{1}{2}B(y,x)\frac{1}{2}\Gamma(x+y) = \frac{\Gamma(x)}{2}\frac{\Gamma(y)}{2}\ ,0 < x,\ 0 < y$$

This completes the proof of the theorem.

Example 24.1 Evaluate $\int_0^1 x^4(1-x)^3 dx$

Solution: $\int_0^1 x^4(1-x)^3 dx = \int_0^1 x^{5-1}(1-x)^{4-1} dx = B(5,4) = \dfrac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \dfrac{1}{280}$

Example 24.2 Evaluate $\int_0^1 \dfrac{1}{\sqrt[3]{x^2(1-x)}} dx$

Solution: $\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}}dx = \int_0^1 x^{\frac{1}{3}-1}(1-x)^{\frac{2}{3}-1}dx = \beta(\frac{1}{3},\frac{2}{3}) = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(1)} = \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$

Example 24.3 Evaluate $\int_0^1 \sqrt{x}.(1-x)dx$

Solution: $\int_0^1 \sqrt{x}(1-x)dx = \int_0^1 x^{\frac{3}{2}-1}(1-x)^{2-1}dx = \beta\left(\frac{3}{2},2\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}$

$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$

Thus $\int_0^1 \sqrt{x}.(1-x)dx = \frac{4}{15}$

Example 24.4 Given $\int_0^\infty \frac{x^{q-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin n\pi}$

Proof: We know,

for $0 < x < \infty, 0 < y < \infty$,

$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$

$$\int_0^\infty \frac{x^{q-1}}{(1+x)^{q+(1-q)}}dx = \beta(q,1-q)$$

$$= \frac{\Gamma(q)\Gamma(1-q)}{\Gamma(1)} = \Gamma(q)\Gamma(1-q)$$

Example 24.5 Evaluate $I = \int_0^\infty \frac{dx}{(1+x^4)}$

Solution: Let $x^4 = t, 4x^3dx = dt$

$$I = \frac{1}{4}\int_0^{\infty}\frac{t^{-\frac{3}{4}}}{1+t}dt = \frac{1}{4}\int_0^{\infty}\frac{t^{\frac{3}{4}-1}}{1+t}dt = \frac{1}{4}\frac{\pi}{sin\frac{1}{4}\pi} = \frac{\pi}{4\times\frac{1}{\sqrt{2}}}$$

Exercises

1. $\int_0^1 t^3\,(1-t)^3 dt$
2. $\int_0^1 \sqrt[5]{t(1-t)}\,dt$
3. $\int_{0+}^1 (1-\frac{1}{t})^{\frac{1}{5}}\,dt$
4. $\int_0^{\frac{\pi}{2}-} \sqrt{\tan x}\,dx$
5. $\int_0^{\frac{\pi}{2}} (\sin 2x)^{\frac{1}{4}}\,dx$
6. $\int_{0+}^{\infty} \frac{1}{\sqrt{t}(1+t)}\,dt$
7. $\int_{0+}^{\infty} \frac{t\,dt}{(1+t)^5}$
8. $\int_{0+}^{\infty} \frac{dt}{(1+t)^2\sqrt{1+(1/t)}}$
9. $\int_{0+}^{\frac{\pi}{2}-} (\sin 2x)^{2t-1}\,dx \qquad 0 < t < \infty$

Keywords: Gamma Function, Beta Function, Polar Coordinate.

References

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Suggested Readings

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