Lesson 23

Gamma Function

23.1 Introduction: We shall define a function known as the gamma function, $\Gamma(x)$ which has the property that $\Gamma(n) = (n-1)!$ for every positive integer n. It may be regarded then a generalization of factorial n to apply to values of the variable which are not integer. The function is defined in terms of an improper integral. This integral cannot be evaluated in terms of the elementary functions. It has great importance in analysis and in applications.

Definition 23.1 The Gamma Function: The gamma function is defined by the improper integral

$$\Gamma(\lambda+1) = \int_0^{+\infty} e^{-t} t^{\lambda}\, dt \text{ ------------- } (23.1)$$

which converges for all $\lambda > -1$

To deduce some of the properties of the gamma function, let us integrate Eq. (23.1) by parts:

$$\int_0^{+\infty} e^{-t} t^{\lambda}\, dt = \lim_{R\to+\infty} \int_0^{R} e^{-t} t^{\lambda}\, dt$$

$$= \lim_{R\to+\infty} \left[-e^{-t} t^{\lambda} \Big|_0^R + \lambda \int_0^R e^{-t} t^{\lambda}\, dt \right]$$

$$= \lim_{R\to+\infty} \left[\frac{-R^{\lambda}}{e^R} + 0 \right] + \lambda \int_0^{+\infty} e^{-t} t^{\lambda-1}\, dt$$

$$= \lambda \int_0^{+\infty} e^{-t} t^{\lambda-1}\, dt$$

i.e. $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$ ------------ (23.2)

If we let $\lambda=0$ in Eq 1. these results

$$\Gamma(1)=\lambda\int_0^\infty e^{-t}\,dt=-e^{-t}\Big|_0^\infty=1$$

Using Eq 23.2, we obtain

$\Gamma(2)=1.\Gamma(1)=1$

$\Gamma(3)=2.\Gamma(2)=2!$

$\Gamma(4)=3.\Gamma(3)=3!$ ------------ (23.3)

The equations above represent another important property of the gamma function. $1+\lambda$ is a positive integer.

$\Gamma(\lambda+1)=\lambda!$ ------------ (23.4)

It is interesting to note that $\Gamma(\lambda)$ is defined for all λ except $\lambda=0,-1,-2,......$by the functional equation $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$; infact, we need to know $\Gamma(\lambda)$ only for $1\le\lambda\le 2$ to compute $\Gamma(\lambda)$ for all real values of λ. Fig 1. Illustrates the graph $\Gamma(\lambda)$

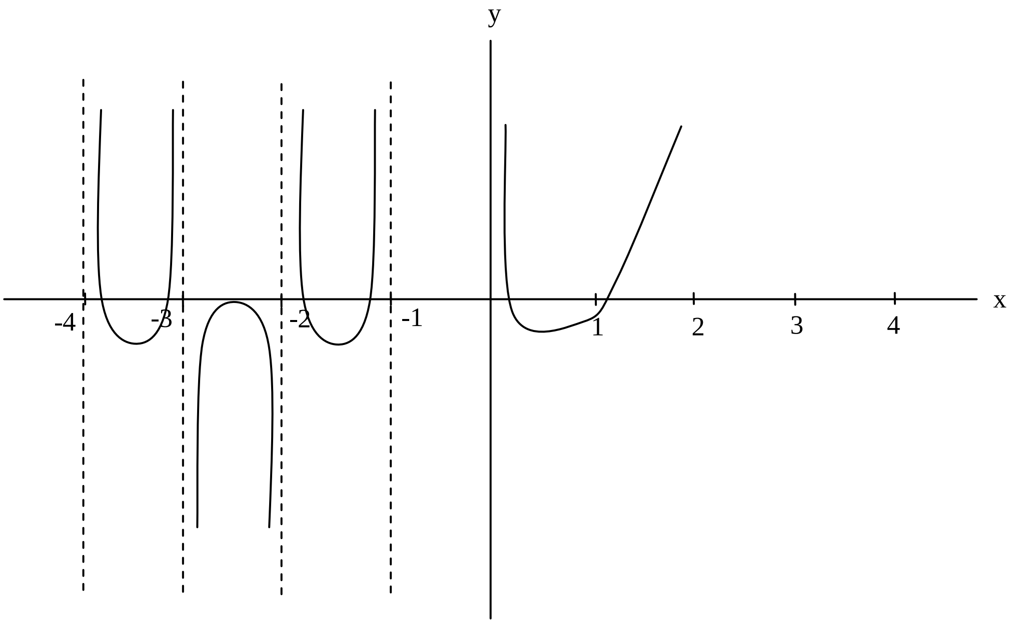

Fig 1. $\Gamma(\lambda)$ the Gamma function

Certain constants related to $\Gamma(x)$. We shall show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. In order to do this, we compute first the so-called probability integral.

Theorem 23.1. $\int_0^{+\infty} e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$

To prove this, consider the double integral of $e^{-x^2-y^2}$ over two circular sectors D_1 and D_2 and the Square S indicated in Fig 2.

Since the integral is positive, we have

$$\iint_{D_1} < \iint_{S} < \iint_{D_2} \qquad (24.1.5)$$

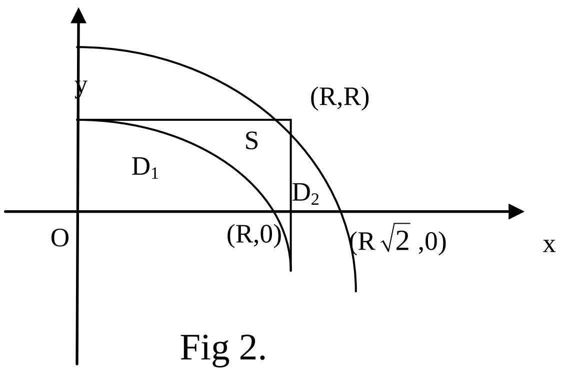

Fig 2.

Now evaluate these integrals by iterate integrals, the centre one in rectangular coordinates, and other two in polar coordinates:

$$\int_0^R e^{-r^2} r\,dr \int_0^{\frac{\pi}{2}} d\theta < \int_0^R e^{-x^2} dx \int_0^R e^{-y^2} dy < \int_0^{R\sqrt{2}} e^{-r^2} r\,dr \int_0^{\frac{\pi}{2}} d\theta$$

$$\frac{\pi}{4}\left(1-e^{-R^2}\right) < \left(\int_0^R e^{-x^2} dx\right)^2 < \frac{\pi}{4}\left(1-e^{-2R^2}\right)$$

Now let $R \to \infty$, then

$$\left(\int_0^{+\infty} e^{-x^2} dx\right)^2 = \frac{\pi}{4}$$

i.e., $\int_0^{+\infty} e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$

Theorem 23.2. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Now, $\Gamma\left(\dfrac{1}{2}\right) = \int_0^{+\infty} e^{-t} t^{\frac{1}{2}-1}\,dt = \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}}\,dt$

$$= 2\int_0^{+\infty} e^{-y^2} dy = \sqrt{\pi} \quad \text{set } t = y^2$$

Example 23.1 Evaluate the integral $\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx$

Solution: Set $x = t^2$, $dx = 2tdt$

$$\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx = 2\int_0^{+\infty} t^{\frac{7}{2}} e^{-t} dt = 2\Gamma(\tfrac{9}{2})$$

From the recursive relation (2) , we obtain

$$\Gamma(\tfrac{9}{2}) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}\Gamma(\tfrac{3}{2}) = \frac{105}{8}\frac{\sqrt{\pi}}{2}$$

Finally, the volume of the integral is

$$\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx = 2 \times \frac{105}{8} \times \frac{\sqrt{\pi}}{2} = \frac{105\sqrt{\pi}}{8}$$

Example 23.2 Express the product
$f(r) = r(r+h)(r+2h).......[r+(n-1)h]$ as a quotient of gamma functions.

Solution: We have

$$f(r) = \left(\frac{r}{h}\right)\left(\frac{r}{h}+1\right)\left(\frac{r}{h}+2\right).....\left(\frac{r}{h}+(n-1)h\right)h^n$$

$$= h^n \frac{\Gamma\left(\frac{r}{h}+1\right)}{\Gamma\left(\frac{r}{h}\right)} \frac{\Gamma\left(\frac{r}{h}+2\right)}{\Gamma\left(\frac{r}{h}+1\right)} .... \frac{\Gamma\left(\frac{r}{h}+n\right)}{\Gamma\left(\frac{r}{h}+n-1\right)}$$

$$= \frac{\Gamma\left(\frac{r}{h}+n\right)}{\Gamma\left(\frac{r}{h}\right)}.h^n$$

obtained by the recursion Eq. 2 with $\lambda = \dfrac{r}{h}$

Some special cases of the result of Example 2 are interesting. For particular case, set r = 1 and h = 2. Then

$$1.3.5....(2n-1) = \frac{2^n\Gamma(n+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2})}$$

But $\frac{1}{2}\Gamma(\tfrac{1}{2}) = \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

Hence

$$1.3.5....(2n-1) = \frac{2^n\Gamma(n+\tfrac{1}{2})}{\sqrt{\pi}}$$

However,

$$1.3.5....(2n-1) = 1.3.5....(2n-1)\frac{2.4.6.....2n}{2.4.6.....2n}$$

$$= \frac{(2n)!}{2^n n!}$$

Now combining the two equations above , we get

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!}{2^n n!} \times \frac{\sqrt{\pi}}{2^n} = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}$$

for n = 1, 2, ……

Other expressions for $\Gamma\left(x\right)$

Theorem 23.3. $\Gamma\left(x\right) = r^x \int_0^{+\infty} e^{-rt}t^{x-1}\,dt,\ r > 0,\ x > 0$

This follows from the definition

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt,\ set\ rt = y$$

Theorem 23.4. $\Gamma(x) = 2\int_0^{+\infty} e^{-t^2} t^{2x-1}\, dt$

Proof: Set $t^2 = y$

Extension of definition

Definition : For n =1,2,….

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2).....(x+n-1)}, -n < x < -n+1$$

Thus we have defined $\Gamma(x)$ for all x except x = 0,-1,-2,…. Observe that when n =1 the right hand side of (6) depends on the values of $\Gamma(x)$ in the interval 0<x<1. It is clear that $\Gamma(x)$ has been defined for negative x in such a way that equation

$$\Gamma(x+1) = x\Gamma(x) \text{for } x \neq 0,-1,-2,.... \text{---------(23.6)}$$

Example 4. Compute $\Gamma\left(\frac{1}{2}\right)$

From equation (7), we have

$$\Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$

i.e., $\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

i.e., $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Exercise

Evaluate each integral

1. $\displaystyle\int_0^{+\infty} \sqrt{x}\, e^{-x} dx$

2. $\displaystyle\int_0^{+\infty} x^2\, e^{-x^2} dx$

3. $\displaystyle\int_0^{+\infty} x^{-4}\, e^{-\sqrt{x}} dx$

4. $\displaystyle\int_0^{+\infty} (1-x)^3\, e^{-\sqrt{x}} dx$

5. $\displaystyle\int_0^{+\infty} x^3\, e^{-\sqrt{x}} dx$

6. Show that the improper integral $\displaystyle\int_0^{+\infty} e^{-t} t^x dt$ converges for $x > -1$ and diverges for $x \le -1$.

7. Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{x \ln\left(\frac{1}{x}\right)}}$

8. Evaluate $\displaystyle\int_0^{\infty} 2^{-9x^2} dx$ using gamma function (Hint : $2^{-9x^2} = e^{-9x^2} \ln 2$)

Ans.: 1. $\Gamma\left(\frac{3}{2}\right)$ *or* $\frac{\sqrt{\pi}}{2}$, 2. 6, 3. ∞, 4. -9394, 5. $2\times 7!$, 6. ,7. $\sqrt{2\pi}$ & 8.

$\frac{1}{6}\sqrt{\frac{\pi}{\ln 2}}$

Keywords: Gamma Function, Convergence of Integral, Factorial Function

References

W. Thomas, Finny (1998). Calculus and Analytic Geometry, 6th Edition, Publishers, Narsa, India.

Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.