Lesson 22

Area & Volume using Double and Triple Integration

22.1 Introduction

We have seen if we take f (x, y) = 1 in the definition of the double integral over a region in Eqn (20.2), is the partial sum reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k,$$

and give area of the region as $n \to \infty$. In that case $\Delta x, \Delta y$ approach zero. In this case we define the area on a rectangular region R to be the limit

$$Area = \lim \sum \Delta A_k = \iint_R dA \qquad (22.1)$$

Example 22.1 Find the area of the region R bounded by y = x and y = x^2 in the first quadrant.

Solution: The area of the region is

$$\int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Bigg|_0^6 = \frac{1}{6}$$

Example 22.2 Find the area of the region R enclosed by the parabola y = x^2 and the line y = x + 2.

Solution: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$$x^2 - 2x + x - 2 = 0 \quad i.e., \quad x(x-2) + 1(x-2) = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1, 2$$

Hence the area $A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx$

$$= \int_{-1}^{2} y\Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} (x + 2 - x^2) dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 2$$

$$= 8 - \frac{16 + 3 + 2}{6}$$

$$= 8 - \frac{7}{2} = \frac{9}{2}$$

Solution:

For order of integration reversed, draw a horizontal lin L_2. It enters at $x = \frac{y}{2}$, leaves at $x = \sqrt{y}$. To include all such lines we let y to n from y = 0 to y = 4. The integral is

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)\,dx\,dy$$

22.1.1 Changing to Polar Coordinates.

When we define the integral of a function f(x, y) over a region R we divide R with rectangles, and their areas easy to compute. But when we work in polar coordinates, however it is more natural to subdivide R into 'polar rectangles' we can find the double integral in polar form as.

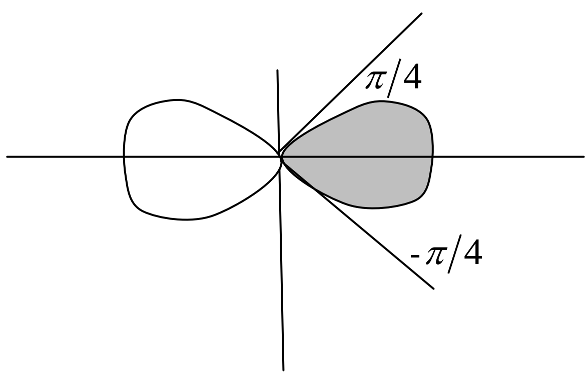
$$\iint F(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$$ ------------ (22.2) , give running numbers.

Where the function $F(r,\theta)$ is defined over a region R bounded by the areas $\theta=\alpha$, $\theta=\beta$ and the continuous curve $r = f_1(\theta)$, $r = f_2(\theta)$.

If $F(r,\theta) \equiv 1$ the constant function whose value is one, then the value over R is the areas of R (which agrees our earlier definition). Thus

Area of R = $\iint\limits_R r\,dr\,d\theta$

Example 22.3 Find the area enclose by the lemniscate $r^2 = 2a^2 \cos 2\theta$.



The area of the right-hand half to be

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\sqrt{2a\cos^2 2\theta}} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left.\frac{r^2}{2}\right|_{r=0}^{r=\sqrt{2a\cos^2 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$$

$$= \left.\frac{a^2}{2}\sin 2\theta\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2}[1-(-1)]$$

$$= a^2$$

The total area is therefore $2a^2$.

22.2 Volume using Triple Integral

If $F(x,y,z) \equiv 1$ is the constant function whose volume is one, then the sums in Eq (1) reduce to $S_n = \sum_{k=1}^{n} 1.\Delta V_K = \sum_{k=1}^{n} \Delta V_K$

As $\Delta_x, \Delta_y, \Delta_z$ all approaches zero, the cells ΔV_k become smaller and we need more cells to fill up D. We therefore define the volume of D to be the triple integral of the constant function f(x, y, z) = 1 over D.

Volume of D = $\lim \sum_{k=1}^{n} \Delta V_k = \iiint_D dV$.

The triple integral Evaluation is hardly evaluated directly from its definition as a limit. Instead, one applies a three-dimensional version of Fubin's theorem to evaluate the integral by repeated single integrations.

22.3 Integrals in Cylindrical and Spherical Coordinates

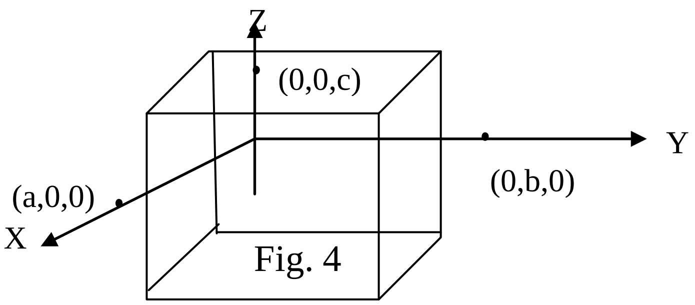


Fig. 4

Fig. 4 shows a system of mutually orthogonal coordinates axes OX, OY, OZ. The Cartesian coordinates of a point P(x, y, z) in the space may be read from the coordinates axes by passing planes through P perpendicular to each axis. The points on the x-axis have their y- and z- ordinates both zero. Points in a plane perpendicular to the z-axis, say, all have the same z - coordinate. Thus of the points in the plane perpendicular to the z- axis and 5 units above the xy-plane all have coordinates of the form (x,y,5) . We can write z = 5 as an evaluation for this plane. The three planes x = 2, y = 3, z = 5 intersect in the point P (2, 3, 5). The points of the yz- plane are obtained setting x = 0. The three coordinates planes x = 0, y = 0, z = 0 divide the space into eight cells, called octants. The octant in which all three coordinates are positive is the first octant, but there is no conventional numbering of the remaining seven octants.

Example 2. Describe the set of points P(x, y, z) whose Cartesian coordinates satisfy the simultaneous equation $x^2 + y^2 = 4, z = 3$.

Solution: The points all horizontal plane z = 3, and in this plane they lie in this cirle $x^2 + y^2 = 4$.Thus we may describe the set of the circle in the plane $x^2 + y^2 = 4$ in the plane z = 3.

22.3.1 Cylindrical Coordinates

It is frequently convenient to use cylindrical coordinates (r, θ, z) to locate a point in space. These are just the polar coordinates (r, θ) used instead of (x, y) in the plane z = 0, coupled with the z- coordinates. Cylindrical and Cartesian coordinate are therefore related by the following equations : Equations relating cartesian and cylindrical coordinates.

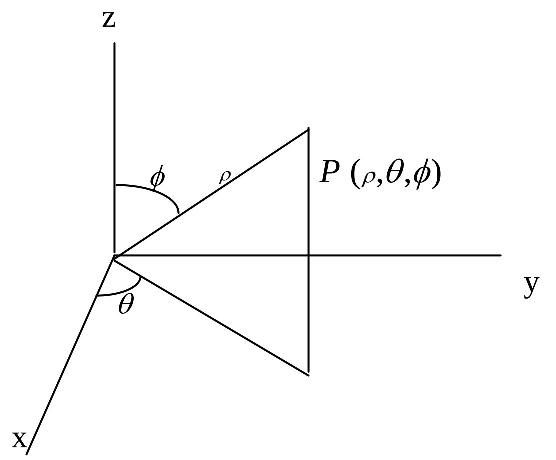
$$x = r\cos\theta \qquad r = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z$$

22.3.2 Spherical Coordinates

Spherical coordinates are useful when there is a center of symmetry that we can take as the origin. The spherical coordinates (ρ, φ, θ) are shown the first coordinates $\varphi = |OP|$ is the distance from the origin to the point. It is never negative. The equation $\varphi =$ constant describes the surface of the sphere of radius φ with centre O.



The second spherical coordinate ϕ, is the angle measured down from the z-axis to the line OP. The equation $\rho =$constant describes cone with vertex at O, axis OZ and generating angle ϕ, provide we broaden our interpretation of the word "cone" to include the xy- plane for which $\phi = \frac{\pi}{2}$ and cones the generation angles greater than $\frac{\pi}{2}$.

The third spherical coordinates θ is the same as the angle θ in cylindrical coordinates, namely, the angle from the xz-plane the plane through P and the z-axis.

22.3.3. Coordinate Conversion Formulas

We have the following relationships between these Cartesian (x, y, z), cylindrical (r, θ, z), *and spherical* (ρ, φ, θ)

Polar to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$r = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\theta$

Volume : $\iiint dx\,dy\,dz = \iiint dz\,rdr\,d\theta = \iiint \rho^2 \sin\theta\, d\rho\, d\phi\, d\theta$

Exercises

1. Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 1$

2. Find the area of the region R bounded by $y = x$ and $x = y^2$ in the first quadrant.

3. Find the volume of the solid in the first octant bounded by the paraboloid. $z = 36 - 4x^2 - 9y^2$

4. Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = 9^2$ and $x^2 + z^2 = 9^2$.

5. The volumes of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.

6. The volume in the first octant bounded by the planes $x + z = 1$, $y + 2z = 2$.

7. The volume of the wedge cut from the cylinder $x^2 + y^2 = 1$ and the plane $z = y$ above and plane below.

8. The volume of the region in the first octant bounded by the coordinate planes, above by the cylinder $x^2 + z = 1$ and on the right by the paraboloid $y = x^2 + z^2$

(Hint: Integrate first with respect to y)

Ans.: 1. ,2. , 3. 27π, 4. $\frac{16a^3}{3}$, 5. $\frac{1}{6}|abc|$, 6. $\frac{2}{3}$, 7. $\frac{2}{3}$ & 8. $\frac{2}{7}$

Keywords: Area, Volume, Double Integral, Triple Integral

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Suggested Readings

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