Lesson-21

Triple Integration

21.1 Introduction

If F(x, y, z) is the function defined on a bounded region D in space (a solid ball or truncated cone, for example of something resembling a swiss cheese, or a finite union of such objects) then the integral of F over D defined in the following way.

We partition a rectangular region about D into rectangular cells by planes parallel to the co-ordinate planes, as shown in Fig.

The cells have dimensions Δx by Δy by Δz. We number the cells that lie inside D in some order $\Delta V_1, \Delta V_2, \ldots\ldots, \Delta V_n$,
choose a point (x_k, y_k, z_k) in each ΔV_k, and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k \text{ ------------- (21.1)}$$

If F is continuous and the bounding surface of D is made of smooth surfaces joined along continuous curves, then as $\Delta x, \Delta y$ and Δz all approach zero the sum S_n will approach all limit.

$$\lim S_n = \iiint_D F(x, y, z)\, dV$$

We call this limit the triple integral of F over D. The limit also exists for some discontinuous functions.

Triple integrals share many algebraic properties with double and single integrals. Writing F by $F(x, y, z)$ and G for $G(x, y, z)$, we have the following

1. $\iiint\limits_D k\,F\,dV = k\iiint\limits_V F\,dV \;\; (any\,number\,k)$
2. $\iiint\limits_D (F \pm G)dV = \iiint\limits_D F\,dV \pm \iiint\limits_D G\,dV$
3. $\iiint\limits_D F\,dV \geq 0 \; if \;\; F \geq 0 \; in \; D$
4. $\iiint\limits_D F\,dV \geq \iiint\limits_D G\,dV \;\; if \;\; F \geq G \; on \; D$

If the domain D of a continuous function F is partitioned by smooth surface into a finite number of cells $D_1, D_2, ..., D_n$, then

5. $\iiint\limits_D F\,dV = \iiint\limits_{D_1} F\,dV + \iiint\limits_{D_2} F\,dV + ... + \iiint\limits_{D_n} F\,dV$

The triple integral Evaluation is hardly evaluated directly from its definition as a limit. Instead, one applies a three-dimensional version of Fubin's theorem to evaluate the integral by repeated single integrations.

For example, suppose we want to integrate a continuous function F(x, y, z) over a region D that is bounded below by a surface z = f_1(x, y) above by the surface
z = f_2(x, y) , and on the side by a cylinder C parallel to the z – axis (Fig. 2). Let R denote the vertical projection of D onto the xy-plane enclosed by C. The integral of F over D is then evaluated as

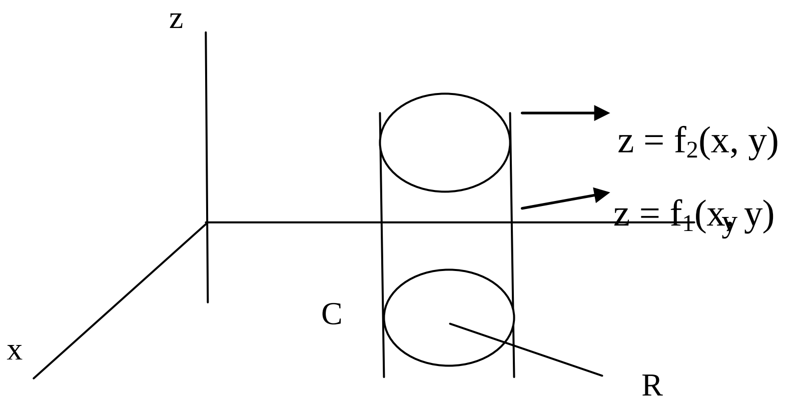

Fig.12

$$\iiint_D F(x,y,z)dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz \right) dy\,dx$$

or $\iiint_D F(x,y,z)dV = \iint_R \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz\,dy\,dx$ -------------- (21.1)

If we omit the parenthesis .The z-limits of integration indicate that for every (x, y) in the region R, z may extend from the lower surface $z = f_1(x, y)$ to the upper surface $z = f_2(x, y)$. The y – and x - limits of integration have not given explicitly in Eq (21.1) but are to be determined in the usual way from the boundaries of R.

We will find the equation of the boundary of R by eliminating z between the two equations $z = f_1(x, y)$ and $z = f_2(x, y)$. This gives

$$f_2(x, y) = f_1(x, y),$$

an equation that contains no z and that defines the boundary of R in the xy - plane.

To give the z -limits of integration in any particular instance we may use a procedure like the one for double integrals. We imagine a line L through a point (x, y) in R and parallel to the z-axis. As z increases, the line enters D at $z = f_1(x, y)$ and leaves D at $z = f_2(x, y)$. These give the lower and upper limits of the integration with respect to z . The result of this integration is now a function of x and y alone, which we integrate over R, giving limits in the familiar way.

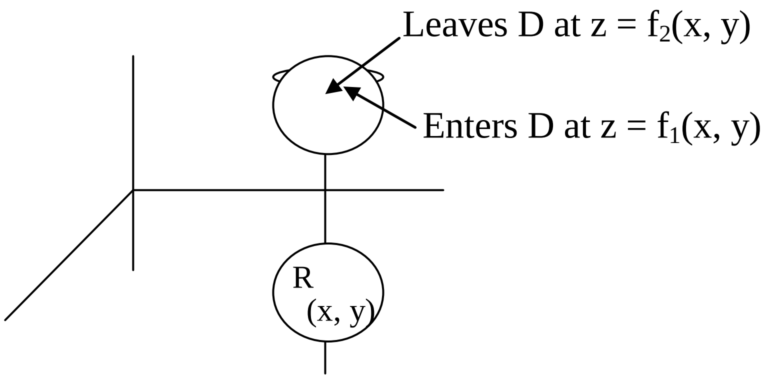

Fig. 12

Example 21.1 Find the volume enclosed between the two surfaces $z = x^2+3y^2$ and
$z = 8-x^2-y^2$.

Solution: The two surfaces intersect on the surface

$$x^2+3y^2 = 8-x^2-y^2$$

$$\text{or} \qquad x^2+2y^2 = 4$$

which is elliptic .

So the volume of the surface is

$$V=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left(8-x^2-y^2-x^2-3y^2\right)dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left(8-2x^2-4y^2\right)dy\,dx$$

$$=\int_{-2}^{2}\int_{0}^{\sqrt{(4-x^2)/2}}2\left(8-2x^2-4y^2\right)dy\,dx$$

$$=\int_{-2}^{2}2\left((8-2x^2)y-\frac{4}{3}y^3\right)\Bigg|_{0}^{\sqrt{(4-x^2)/2}}dx$$

$$=\int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{(4-x^2)}{2}}-\frac{8}{3}\left(\frac{(4-x^2)}{2}\right)^{\frac{3}{2}}\right)dx$$

$$=\frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{\frac{3}{2}}dx$$

$$=\frac{8\sqrt{2}}{3}\int_{0}^{2}\left(4-x^2\right)^{\frac{3}{2}}dx$$

$$=8\pi\sqrt{2}$$

As we know, there are sometimes two different orders in which the single integrations that evaluate a double integral may be worked (but not always). For triple integral there are sometimes (but not always) as many as six workable orders of integration. The next example shows an extreme case in which all six are possible.

Example 21.2 Each of the following integrals gives the volume of the solid shown in Fig 3.

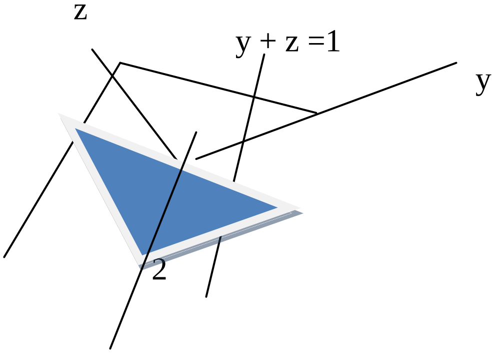

Fig 3.

$$(a)\int_0^1\int_0^1\int_0^2 dx\,dy\,dz \qquad (b)\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy$$

$$(c)\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz \qquad (d)\int_0^2\int_0^1\int_0^{1-z} dy\,dz\,dx$$

$$(e)\int_0^1\int_0^2\int_0^{1-y} dz\,dx\,dy \qquad (f)\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx$$

<u>EXERCISES</u>

1. Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the co-ordinate planes and the planes x = 1, y = 2,
 z = 3. Evaluate one of the integrals.

2. Write six different intersected triple integrals of the volume in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane y = 3. Evaluate one of the integrals.

3. Write an iterated triple integrals in the order dz dy dx for the volume of the region bounded below by the xy-plane and above by the paraboloid $z = x^2 + y^2$ and lying inside the cylinder $x^2 + y^2 = 4$.

4. Rewrite the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$ as an equivalent integrated integral in the order.

a) *dy dz dx* b) *dy dx dz* c) *dx dy dz* d) *dx dz dy* e) *dz dx dy*

Ans.: **1.** $\int_0^1\int_0^2\int_0^3 dz\,dy\,dx$, $\int_0^2\int_0^1\int_0^3 dz\,dx\,dy$, $\int_0^3\int_0^2\int_0^1 dx\,dy\,dz$, $\int_0^3\int_0^1\int_0^2 dy\,dx\,dz$, $\int_0^2\int_0^3\int_0^1 dx\,dz\,dy$, the value of each integral is 3, 2. $\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy$, $\int_0^2\int_0^3\int_0^{\sqrt{4-x^2}} dz\,dy\,dx$, $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 dy\,dz\,dx$,

$\int_0^2\int_0^{\sqrt{1-z^2}}\int_0^3 dy\,dz\,dx$, $\int_0^3\int_0^2\int_0^{\sqrt{4-z^2}} dx\,dz\,dy$, $\int_0^2\int_0^3\int_0^{\sqrt{1-x^2}} dx\,dy\,dz$. Value of each integral is 12π .

3. $4\int_0^2\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} dz\,dy\,dx$ & 4.

Keywords: Triple integral, Fubini’s theorem, volume

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Suggested Readings

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