Lesson 20

Double Integration

20.1 Introduction

In applications of calculus we have seen with integrals of functions of a single variable. The integral of a function y = f(x) over an interval [a, b] is the limit of approximating sums

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \text{ ------------ (20.1)}$$

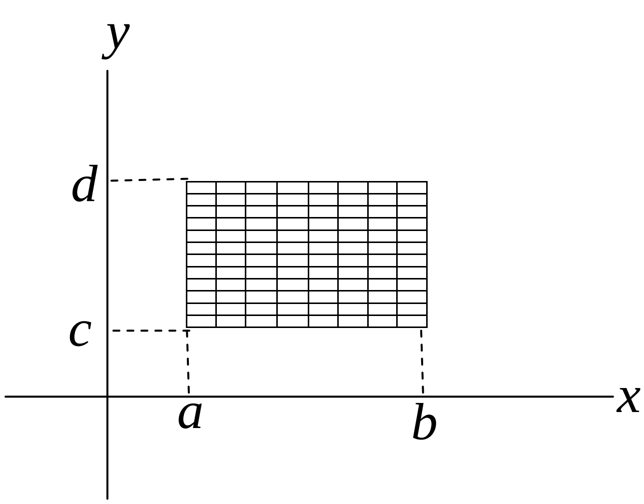
Where $a = x_0 \le x_1 \le x_2 \le ..... \le x_n = b,\ \Delta x_k = x_{k+1} - x_k$ and c_k is the any point from the interval $[x_k, x_{k+1}]$. The limit in (20.1) is taken as the length of the longest subinterval approaches zero. The limit is guaranteed to exist if f is continuous and also exists when f is bounded and has only finitely many points of discontinuity in [a, b] . There is no loss in assuming the intervals $[x_k, x_{k+1}]$ to have common length $\Delta x = \frac{b-a}{n}$, and limit may thus obtain by letting $\Delta x = 0$ as $n \to \infty$.

If f(x) > 0, then $\int_a^b f(x)dx$ from x = a and x = b, but in general the integral has many other important interpretations (distance, volume, arc length, surface area, moment of inertia, mass, hydrostatic pressure, work) depending on the nature and interpretation of f.

In this Lesson we shall see that integrals of functions of two or more variables which are called multiple integrals and defined I much the same way as integrals of functions of single variable.

Double Integrals: Here we define the integral of a function f(x, y) of two variables over a rectangular region in xy-plane. We then show how such an integral is evaluated and generalize the definition to include bounded regions of a more general nature.

Double Integrals over Rectangles:



Suppose that f(x, y) is defined on a rectangular region R defined by

$$R : a \le x \le b,\ c \le y \le d$$

(see the figure 1.)

We imagine R to be covered by a network of lines parallel to x-axis and y-axis, as shown in Fig 1. These lines divide R into small pieces of area

$$\Delta A = \Delta x \Delta y$$

We number these in some order

$$\Delta A_1, \Delta A_2, ..., \Delta A_n,$$

Choose a point (x_k, y_k) in each piece of ΔA_k and from the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k \text{ ------------------- (20.2)}$$

If f is continuous throughout R, then we define mesh width to make both Δx and Δy go to zero the sums in (2) approach a limit called the double integral of f over R that is denoted by $\iint_R f(x, y)dA \ \ or \ \ \iint_R f(x, y)dx\,dy$

Thus $\iint_R f(x, y)dA = \lim_{\Delta A \to 0} \sum_{1}^{n} f(x_k, y_k)\Delta A_k$ ---------- (20.3)

As with functions of a single variable, the sums approach this limit no matter how the interval [a, b] and [c, d] that determine R are subdivide, along as the lengths of the subdivisions both go to zero. The limit (20.3) is independent of the order in which the area ΔA_k are numbered, and independent of the choice of (x_k, y_k) within each ΔA_k. The continuity of f sufficient condition or the existence of the double integral, but not a necessary one, and limit question exists for many discontinuous functions also.

20.1.1 Properties of Double Integral

Like "single" integrals, we have the following properties for double integrals of continuous functions which are useful in computations and applications.

(i) $\iint_R k\, f(x,y)dA = k\iint_R f(x,y)dA$ (any number k)

(ii) $\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$

(iii) $\iint_R [f(x,y)-g(x,y)]dA = \iint_R f(x,y)dA - \iint_R g(x,y)dA$

(iv) $\iint_R f(x,y)dA \geq 0 \; if \;\; f(x,y) \geq 0 \; on \; R$

(v) $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA \; if \;\; f(x,y) \geq g(x,y) on \; R$

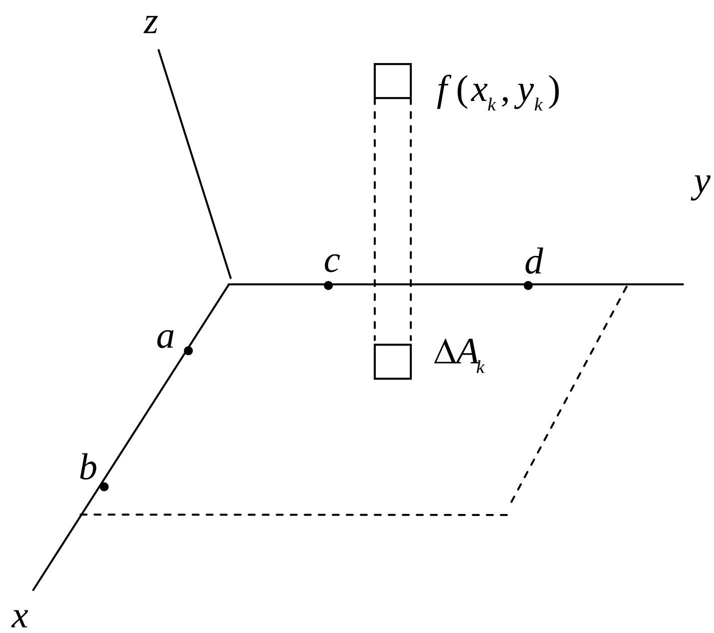
(vi) If $R = R_1 \cup R_2$, $R_1 \cap R_2$, R is the union of two non-overlapping rectangles R_1 and R_2, we have

$$\iint_{R_1 \cup R_2} f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

Volume: When f(x, y) > 0, we may interpret $\iint_R f(x,y)dA$ as the volume of the solid enclosed by R, the planes x = a, x = b, y = c, y = d , and the surface z = f(x, y) see fig 2.

Each term $f(x_k, y_k)\Delta A_k$ in the sum

$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular prism y that



approximate the volume of the portion of the solid that stands above the box - ΔA_k. The sum S_n thus approximates what we call the total volume of the solid, and we define this volume to be

Volume = lim S_n = $\iint_R f(x, y)dA$

20.1.2 Fubbin's theorem for calculating double integrals:

Theorem 20.1. (Fubbin's theorem (1st form))

If f(x, y) is continuous on the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

Fubbin's theorem shows that double integrals over rectangles can be calculated as iterated integrals. This means that we can evaluate a double integral by integrating one variable at a time, using the integration techniques we already know for function of a single variable.

Fubin's theorem also says that we may calculate the double integral by integrating in either order (a genuine convenience). In particular, when we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to y-axis. We get same answer either way.

Even more important is the fact that Fubin's theorem holds for any continuous function f(x, y). In particular it may have negative values as well as positive values on R, and the integrals we calculate with Fubin's theorem may represent other things besides volumes.

Example 20.1: Suppose we wish to calculate the volume under the plane z = 4-x-y over the region $R: 0 \le x \le 2,\ 0 \le y \le 1$ in the xy – plane.

Solution: The volume under the plane is given by $\iint_R (4 - x - y)\, dA$.

Next we have to calculate the double integral.

Now we will complete the stated example.

$$\iint_R f(x, y)\, dA = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy$$

$$= \int_0^1 \left(4x - \frac{x^2}{2} - xy \right)\Bigg|_0^2 dy$$

$$= \int_0^1 (8 - 2 - 2y)\, dy$$

$$= \int_0^1 (6 - 2y)\, dy$$

$$= 6y - y^2 \Big|_0^1 = 5$$

Example 20.2 Calculate $\iint_R f(x, y) dA$ for

$f(x, y) = 1 - 6x^2 y$ and $R: 0 \le x \le 2,\ -1 \le y \le 1$

Solution: By Fubin's theorem

$$\iint_R f(x, y) dA = \int_{-1}^{1} \int_0^2 (1 - 6x^2 y) dx\, dy$$

$$= \int_{-1}^{1} \left(x - 2x^3 y \right)\Big|_{-1}^{2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy$$

$$= 2y - 8y^2 \Big|_{-1}^{1}$$

$$= (2 - 8) - (-2 - 8) = 4$$

Reversing the order of integration gives the same answer:

$$\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dy\,dx = \int_{0}^{2} y-3x^2y^2\Big|_{-1}^{1}\,dx$$

$$= \int_{0}^{2}\left[(1-3x^2)-(-1-3x^2)\right]dx$$

$$= \int_{0}^{2}\left[1-3y^2+1+3x^2\right]dx$$

$$= 2x\Big|_{0}^{2} = 4\,.$$

20.1.2 How to determine the limits of Integration

The difficult part of evaluating a double integral can be finding the limits of integration. But there is a procedure to follow:

If we want to evaluate over a region R, integrating first with respect to y and then with respect to x, we take the following steps:

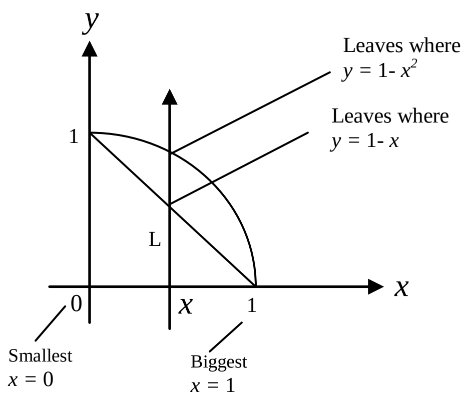
1. We imagine a vertical Line L cutting through in the direction of increasing y

2. We integrate from the y-value where L enters R to the y-value where L leaves R

3. We choose x-limits that include all the vertical lines that pass through R

Example 20.3 Change the order of integral $\int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)dy\,dx$

To calculate the same double integral as an iterated integral with order of integration reversed consider (the figure), by using the above procedure, we have



$$\int_{0}^{1}\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

Example 20.4 Calculate $\iint_A \frac{\sin x}{x}\,dA$ where A is the triangle in the xy-plane bounded by the x-axis, the line y = x and the line y = 1.

Solution: $\int_0^1\left(\int_0^x \frac{\sin x}{x}\,dy\right)dx$

$$=\int_0^1\left(\frac{\sin x}{x}\,y\Big|_{y=0}^{y=x}\right)dx$$

$$=\int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = -\cos 1 + \simeq .46$$

If we reverse the order of integration and try to calculate

$\int_0^1 \int_y^1 \frac{\sin x}{x} dx\, dy$, we can't evaluate it because we can't express $\int \frac{\sin x}{x}$ in terms of elementary functions.

PROBLEM

Evaluate the following integrals and sketch the region over which each integration takes place.

1. $\int_0^3 \int_0^2 (4 - y^2) dy\, dx$

2. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy\, dx$

3. $\int_0^{\pi} \int_0^x x \sin y\, dy\, dx$

4. $\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$

5. Find the value of the integral $\int_{10}^1 \int_0^{\frac{1}{y}} y e^{xy} dx\, dy$

6. Sketch the region of integration of $\int_0^2 \int_{x^2}^{2x} f(x, y) dy\, dx$ and express the integral as an equivalent double integral with order of integration.

Ans.: 1. 16, 2. 0, 3. $\frac{\left(4+\pi^2\right)}{2}$, 4. $\frac{\pi}{4}$, 5. $9-9e$ & 6.

Keywords: Multiple Integrals, Double Integrals, Triple Integrals, Area, Volume

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Suggested Readings

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