Lesson 2

Taylor's theorem / Taylor's expansion, Maclaurin's expansion

2.1 Introduction

In calculus, Taylor's theorem gives us a polynomial which approximates the function in terms of the derivatives of the function. Since the derivatives are usually easy to compute, there is no difficulty in computing these polynomials.

A simple example of Taylor's theorem is the approximation of the exponential function e^x near $x = 0$.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \frac{x^n}{n!}$$

The precise statement of the Taylor's theorem is as follows:

Theorem 2.1: If $n \geq 0$ is an integer and f is a function which is n times continuously differentiable on the closed interval $[a, x]$ and $n+1$ times differentiable on the open interval (a, x), then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+\frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Here, n! denotes the factorial of n, and $R_n(x)$ is a remainder term, denoting the difference between the Taylor polynomial of degree n and the original function. The remainder term $R_n(x)$ depends on x and is small if x is close enough to a.

Several expressions are available for it. The Lagrange form is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = \mathrm{a} + \theta(x-a)$$

where $0 < \theta < 1$.

If we put $a = 0$, Taylor's formula reduces to Maclaurin's formula.

where ξ lies between a and x.

Notes

- In fact, the mean value theorem is used to prove Taylor's theorem with the Lagrange remainder term.
- The Taylor series of a real function $f(x)$ that is infinitely differentiable in a neighborhood of a real number a, is the power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- In general, a function need not be equal to its Taylor series, since it is possible that the Taylor series does not converge, or that it converges to a different function.
- However, for some functions $f(x)$, one can show that the remainder term $R_n(x)$ approaches zero as n approaches ∞. Those functions can be expressed as a Taylor series in a neighbourhood of the point a and are called analytic.

Example 2.1 Show that $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Solution:

Here $f(x) = \sinh x$, $f'(x) = \cosh x$, So

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$f(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$R_n(x) = \frac{h^n}{n!}f^{(n)}(a + \theta h)$. But for $a = 0$ and $h = x$

$$|R_n(x)| = |\frac{x^n}{n!}f^{(n)}(\theta x)|$$

$$\lim_{n\to\infty}|R_n| = \lim_{n\to\infty}|\frac{x^n}{n!}||cosh(\theta x)| = 0$$

Example 2.2 . Find the Taylor series expansion of $\frac{1}{x^2-4}$

Solution: $f(x) = \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

$$= \frac{A}{x+2} + \frac{B}{x-2}$$

$$= -\frac{1}{4(x+2)} + \frac{1}{4(x-2)}$$

$$= -\frac{1}{8(1+\frac{x}{2})} + \frac{1}{-8(1-\frac{x}{2})}$$

$$= -\frac{1}{8}(1+\frac{x}{2})^{-1} - \frac{1}{8}(1-\frac{x}{2})^{-1}$$

for $|\frac{x}{2}| < 1$, we have

$$= -\frac{1}{8}[1-\frac{x}{2}+(-\frac{x}{2})^2+(-\frac{x}{2})^3\cdots]$$

$$-\frac{1}{8}[1+\frac{x}{2}+(\frac{x}{2})^2+(\frac{x}{2})^3\cdots]$$

$$= -\frac{1}{8}[2 + (\frac{x}{2})^2 + \cdots]$$

Example 2.3 : Find $f^{(100)}(0)$ if $f(x) = e^{x^2}$

Ans: $f^{(100)}(0) = \frac{100!}{50!}$.

Questions: Answer the following questions.

1. Expand in power of $x-2$ of the polynomial $x^4 - 5x^3 + 5x^2 + x + 2$.
2. Expand in power of $x+1$ of the polynomial $x^5 + 2x^4 - x^2 + x + 1$.
3. Write Taylor’s formula for the function $y = \sqrt{x}$ when $a = 1, n = 3$.
4. Write the Maclaurin formula for the function $y = \sqrt{1+x}$ when $n = 2$.
5. Using the results of above problem, estimate the error of the approximate equation $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ when $x = 0.2$.
6. Write down the Taylor’s expansion for the function $f(x) = \sin x$ about the point $a = \frac{\pi}{4}$ with $n = 4$.
7. Applying Taylor’s theorem with remainder prove that $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$.
8. Applying Maclaurin’s theorem with remainder expand

   (i) $\ln(1+x)$ (ii) $(1+x)^m$.

Keywords: Taylor's Formula, Taylor's Series, Maclaurin Formula and Series.

References

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Suggested Readings

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