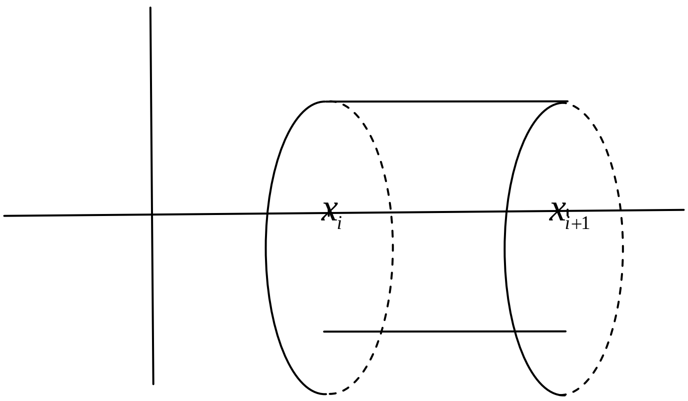
Lesson 19

Volume and Surface of Revolution

19.1 Introduction

Volume of Revolution: We start our applications with volumes of revolutions. Our aim is to find the lengths, areas and volumes of the standard geometric figures.

Let $y = f(x)$ be continuous function of x on the interval with $[a,b]$ with $(a < b)$. Assume that $f(x) \geq 0 \ \forall \ x \in [a,b]$. If we revolve $y = f(x)$ around axis, we obtain a solid, whose volume we want to compute.



Take a partition of $[a,b]$ say $a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq .....x_n \leq= b$

Let c_i be a minimum of f on the interval $[x_i, x_{i+1}]$ and d_i be the maximum of f in that interval. Then the solid of revolutions is that small interval lies between a small cylinder and a big cylinder. The width of these cylinders is $x_{i+1} - x_i$ and the radius is $f(c_i)$ for the small cylinders and $f(d_i)$ for the big cylinder. Hence the volume of revolutions, denoted by V satisfies the inequalities

$$\sum_{i=0}^{n-1} \pi f(c_i)^2 (x_{i+1} - x_i) \le V \le \sum_{i=0}^{n-1} \pi f(d_i)^2 (x_{i+1} - x_i)$$

It is therefore reasonable to define this volume to be $V = \int_a^b \pi f(x)^2 \, dx$

If we revolve the curve around $x = \phi(y)$ around $y-axis$ and $\phi(y) \ge 0 \ \forall \ y \in [c,d]$, we define the volume to be $V = \int_c^d \pi f(y)^2 \, dy$

If the curve be expressed by $x = f(t),\ y = \phi(t)$ $V = \pi \int_a^b y^2 \, dx = \pi \int_{t_1}^{t_2} (\phi(t))^2 \ f'(t) \, dt$ where t_1, t_2 are values of t that corresponds to $x = a$ and $x = b$ respectively.

Example 19.1: Compute the volume of the sphere of radius 1.

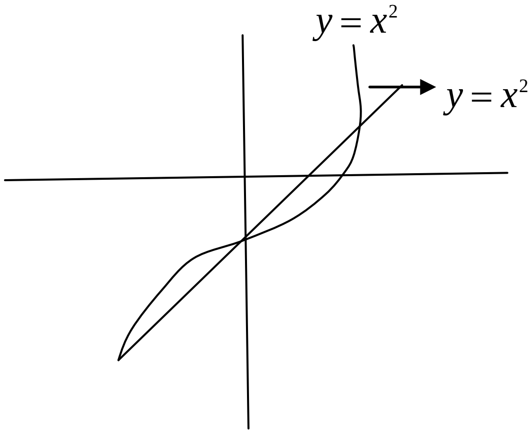
Solution:

We take the function $y = \sqrt{1 - x^2}$ between 0 and 1. If we rotate this curve around $x-axis$, we shall get half the sphere. Its volume is therefore

$$\int_0^1 \pi(1 - x^2) dx = \pi(x - \frac{x^3}{3})\Big|_0^1 = \frac{2}{3}\pi$$

So the volume of full sphere is $2 \times \frac{2}{3}\pi = \frac{4}{3}\pi$

Example 19.2: Find the volume obtained by rotating the region between $y = x^3$ and $y = x$ in the first quadrant around the $x-axis$.



The graph of the region is given on the figure.

As $x^3 = x \Rightarrow x(x^2 - x) = 0 \Rightarrow x = 0, x = \pm 1$, for first quadrant we take $0 \le x \le 1$. The required V volume is equal to the difference of the volume obtained by rotating $y = x$ and $y = x^2$.

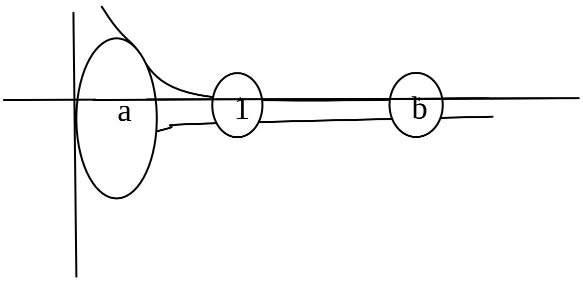
Let $f(x) = x, g(x) = x^3$. Then

$$V = \pi \int_0^1 f(x)^2 dx - \pi \int_0^1 g(x)^2 dx$$

$$= \pi \int_0^1 x^2 dx - \pi \int_0^1 x^6 dx$$

$$= \frac{\pi}{3} - \frac{\pi}{7}$$

Example 19.3: (Volume of Chimneys) . Consider the function $f(x) = \dfrac{1}{\sqrt{x}}$.



Let $0 < a < 1$. The volume of revolution of the curve $y = \frac{1}{\sqrt{x}}$ between $x = a$ and $x = 1$ is given by $\int_a^1 \pi \frac{dx}{x} = \pi \ln x\big|_a^1 = -\pi \ln a$,

As $a \to 0$, $\ln a$ becomes very large negative, so that $-\ln a$ becomes very large positive, and the volume becomes arbitrary large. The above figure illustrates the chimney.

In this computation, we determined the volume of a chimney near the $y-axis$. We can also fixed the volume of the chimney going off to the right, say between 1 and a number b >1. Suppose the chimney is defined by $y = \frac{1}{\sqrt{x}}$. The volume of revolution between 1 and b is given by the integral $\int_1^b \pi \left(\frac{1}{x}\right) dx = \int_0^b \pi \frac{dx}{x} = \pi \ln b$,as $b \to \infty$ we see that this volume becomes arbitrary large (divergent integral)

But we are interested to find finite volume for the infinite chimney.

Example 19.4: Compute the volume of revolution of the curve $y = \frac{1}{x^4}$ between a and 1. Find the limit as $a \to 0$

Solution:

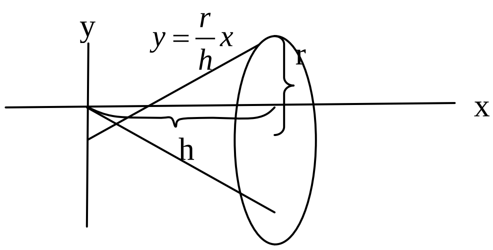
The volume of revolution of the curve $y = \frac{1}{x^4}$ between $x = a$ and $x = 1$

is given by the integral $\int_a^1 \pi \frac{1}{x^{\frac{1}{2}}} dx = \pi \int_a^1 x^{-\frac{1}{2}} dx = \pi \times 2 x^{\frac{1}{2}} \Big|_a^1 = 2\pi\left[1 - \sqrt{a}\right]$

When $a \to 0$ limit becomes 2π

Example 19.5 Find the volume of a cone whose base has a radius r , and a height h , by rotating a straight line passing through the origin around the $x-axis$

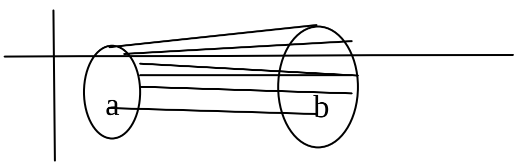
Solution:



The equation of the straight line is $y=\frac{r}{h}x$. Slant height is $y=\frac{1}{x^2}$. Hence the volume of the cone is $\int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \pi\frac{r^2}{h^2}\int_0^h x^2 dx = \frac{\pi r^2}{h^2}\times\frac{h^3}{3} = \frac{1}{3}\pi r^2 h$

19.2 Surface of Revolution

Let $y = f(x)$ be a positive continuously differentiable function on an interval [a,b]. We wish to find a formula for the area of the surface of revolution of the graph of f around the $x-axis$, as given in the figure

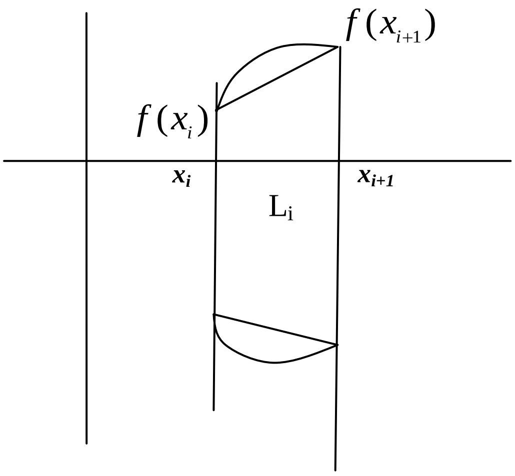


We shall see that the surface area is given by the integral

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

The idea again is to approximate the curve by line segments. We use a partition

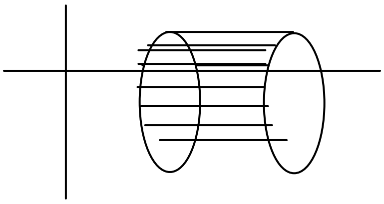
$a=x_0\le x_1\le x_2\le x_3.......\le x_n=b$



On the small interval $[x_i,x_{i+1}]$ the curve is approximated by the line segment joining the points $\left(x_i, f(x_i)\right) and \left(x_{i+1}, f(x_{i+1})\right)$. Let L_i be the length of the segment.

Then $L_i=\sqrt{\left(x_{i+1}-x_i\right)^2+\left(f(x_{i+1})^2-f(x_i)\right)^2}$

The length of a circle of radius y is $2\pi y$. If we rotate the line segment about the then the $x-axis$ area of the surface of rotation will be between $2\pi f(t_i)L_i$ and $2\pi f(s_i)L_i$ where $f(t_i)$ and $f(s_i)$ are the minimum and maximum of f , respectively on the interval $[x_i,x_{i+1}]$. This is illustrated on Fig 1.



On the other hand, by the mean value theorem we can write

$$f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i)\ ,\ c_i \in (x_i, x_{i+1})$$

Hence $L_i = \sqrt{(x_{i+1} - x_i)^2 + f(c_i)^2 (x_{i+1} - x_i)^2}$

$$= \sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$$

Therefore the expression $2\pi f(c_i)\sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$

is an approximation of the surface of revolution of the curve over the small interval $[x_i, x_{i+1}]$.

Now take the sum $\sum_{i=0}^{n-1} 2\pi f(c_i)\sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$

This is a Riemann sum, between the upper and lower sums for the integral

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

Thus it is reasonable that the surface area should be defined by this integral, as was to be shown.

19.2.1 Area of revolution for parametric curves given in parametric form.

Suppose that

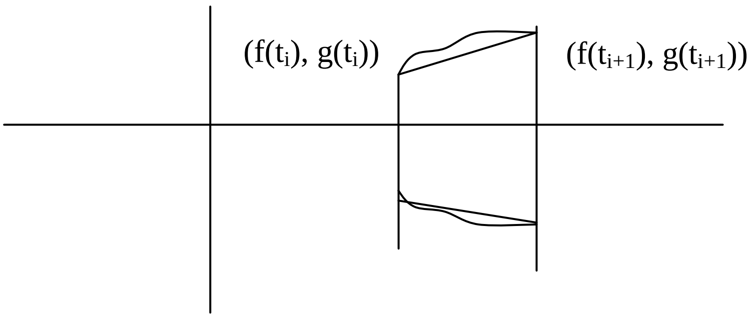
$x = f(t),\ y = g(t),\ a \le t \le b$

We take a partition $a = t_0 \le t_1 \le t_2 \le t_3 .... \le t_n = b$

Then the length of L_i between $\left(f(t_i), g(t_i)\right)$ and $\left(f(t_{i+1}), g(t_{i+1})\right)$ is given by

$$
\begin{aligned}
L_i &= \sqrt{\left(f(t_{i+1}) - f(t_i)\right)^2 + \left(g(t_{i+1}) - g(t_i)\right)^2} \\
&= \sqrt{f'(c_i)^2 + g'(d_i)^2}\,\left(t_{i+1} - t_i\right)
\end{aligned}
$$

where c_i, d_i are numbers between t_i and t_{i+1}



Hence $2\pi g(c_i)\sqrt{f'(c_i)^2 + g'(d_i)^2}\,\left(t_{i+1} - t_i\right)$ is an approximation for the surface of revolution of the curve in the small interval $\left[t_i, t_{i+1}\right]$. Consequently, it is reasonable that the surface of revolution is given by the integral

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

when $t = x$, this coincides with the formula found previously. It is also useful to write this formula symbolically $S = \int 2\pi y ds$

where symbolically, we had used

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

When using this symbolic notation, we don not put limits of integration. Only when we use explicit parameter over an interval $a \le t \le b$ we explicitly write the surface area as

$$S = \int_a^b 2\pi y \frac{ds}{dt} dt$$

Example 19.6 We wish to find the area of a sphere for radius $r > 0$.

Solution: we can view the sphere as the area of revolution of a circle for radius r, and to express the circle in parametric form,

$$x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi$$

Then the formula gives

$$S = \int_0^\pi 2\pi r \sin\theta \sqrt{r^2 \sin\theta + r^2 \cos\theta}\, d\theta$$

$$= \int_0^\pi 2\pi r^2 \sin\theta\, d\theta$$

$$= 2\pi r^2 (-\cos\theta)\Big|_0^\pi$$

$$= 4\pi r^2$$

Exercises

1. Find the volume of sphere of radius r.

Find the volumes of revolution of the following:

2. $y = \frac{1}{\cos x}$ between $x = 0$ and $x = \frac{\pi}{4}$

3. $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

4. The region between $y = x^2$ and $y = 5x$

5. $y = xe^{\frac{x}{2}}$ between $x = 0$ and $x = 1$

6. Compute the volume of revolution of the curve $y = \frac{1}{x^2}$ between $x = 2$ and $x = b$ for any b>2. Does this volume approach a limit as $b \to \infty$? If yes, what limit ?

Ans.: 1. $\frac{4}{3}\pi r^3$, 2. π, 3. $\frac{\pi^2}{8} - \frac{\pi}{4}$, 4. $\frac{2.5^4\pi}{3}$, 5. $\pi(e-2)$ & 6. $\frac{\pi}{24} - \frac{\pi}{3b^3}$, yes: $\frac{\pi}{24}$

Keywords: Lengths, area, volume, surface revolution, volume of chimneys

References

W. Thomas, Finny (1998). Calculus and Analytic Geometry, 6th Edition, Publishers, Narsa, India.

Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.