Lesson 18

Rectification

18.1 Introduction

The method of finding the length of the arc of the curve of is called the rectification. Let $y = f(x)$ be a differentiable function defined on $[a,b]$ with $a < b$ and assume that its derivative is continuous. Our aim is to determine the length of the curve described by the graph. The main idea behind this is to approximate the curve by small line segments and add these up.

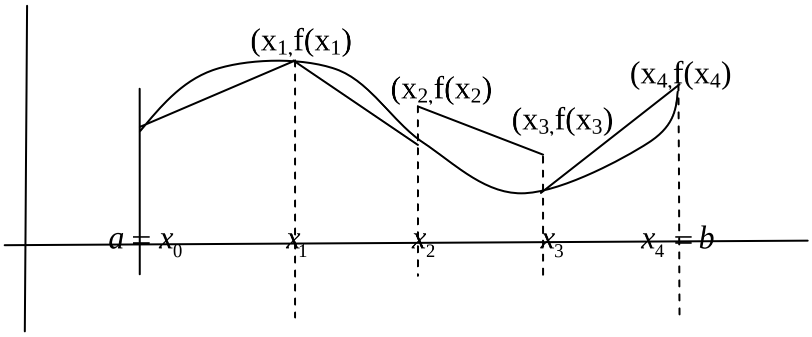


Fig .1

We consider a partition of the interval $[a,b]$. $a = x_0 \le x_1 \le x_2 \le x_3 .... \le x_n = b$

In figure 1 take $n = 4$ for simplification.

For each x_i we have on the curve $(x_i, f(x_i))$. We draw the line segments between two successive points. The length of such a segments the length of the line between

$\left(x_i, f(x_i)\right)$ and $\left(x_{i+1}, f(x_{i+1})\right)$ is equal to $\sqrt{\left(x_{i+1}-x_i\right)^2+\left(f(x_{i+1})-f(x_i)\right)^2}$ ------------- (1)

$\left(f(x_{i+1})-f(x_i)\right)=(x_{i+1}-x_i)\,f'(c_i)$

By mean value theorem, we conclude that

$$f(x_{i+1})-f(x_i)=(x_{i+1}-x_i)\,f'(c_i)\text{ , where } c_i\in(x_i, x_{i+1})$$

Hence (1) becomes now

$$\sqrt{(x_{i+1}-x_i)^2+(x_{i+1}-x_i)^2\,f'(c_i)^2}$$

$$=\sqrt{(x_{i+1}-x_i)^2\left[1+f'(c_i)^2\right]} \text{ -------------- (18.2)}$$

Hence the form of the line segment is

$$\sum_{i=0}^{n-1}\sqrt{1+f'(c_i)^2}\left(x_{i+1}-x_i\right) \text{ ----------------- (18.3)}$$

Now as $f'(x)$ is continuous function. So is $H(x)=\sqrt{1+f'(x)^2}$. So we can write eqn. (3) as $\sum_{i=0}^{n-1}H(c_i)\left(x_{i+1}-x_i\right)$

Since $H(x)$ is continuous on $[a,b]$, $H(c_i)$ satisfies the inequalities:

$$\min_{[x_i, x_{i+1}]} H \le H(c_i) \le \max_{[x_i, x_{i+1}]} H$$

i.e., $H(c_i)$ lies between the minimum and the maximum of on the interval $[x_i, x_{i+1}]$. Thus the sum we have written down lies between a lower sum and an upper sum for the function H . We call such sums as Riemann sums. This is true for every partition of the interval.

We know from basic integration theory that there is exactly one number lying between every upper sum and every lower sum, and that number is the definite interval. Therefore it is reasonable to define:

Length of our curve between a and b

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{\left[1+ f'(x)^2\right]}\, dx \text{ ---------(18.4)}$$

Similarly for $x = \phi(y)$ and $\phi'(y)$ are continuous on $[a,b]$, then the length of our curve between a and b is

$$= \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_a^b \sqrt{1+\phi'(y)^2}\, dy$$

Example18.1 Find the length of the arc of $f(x) = x^{\frac{3}{2}}$ on [0, 4].

Solution:

As f, $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ are both continuous on [0, 4], the length of the arc or length of curve L = $\int_0^4 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$,

Let $1+\frac{9}{4}x = t$, when x=0, t=1,

x=4, t=10

$$\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_1^{10} t^{\frac{1}{2}}dt = \frac{4}{9}\times\frac{2}{3}\times t^{\frac{3}{2}}\Big|_1^{10} = \frac{8}{27}\left[10^{\frac{3}{2}} - 1\right]$$

Example 18.2 Find the length of the curve y=x^2 between x =0 and x =1.

Solution:

From the definition above, we see that the integral is

$$\int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \text{ set u =2x, du = 2dx}$$

When x=0, u=0, x=1, u=2

Hence $\int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du$ --------------------------- (18.5),

We can find the integral $\int_0^b \sqrt{1+x^2}\,dx$ for b > 0, as

$$\frac{1}{4}\left[\frac{1}{2}\left(b+\sqrt{b^2+1}\right)^2 + 2\ln\left(b+\sqrt{b^2+1}\right) - \frac{1}{2}\left(b+\sqrt{b^2+1}\right)^{-2}\right]$$

So $\int_0^2 \sqrt{1+u^2}\,du = \frac{1}{4}\left[\frac{1}{2}\left(2+\sqrt{5}\right)^2 + 2\ln\left(2+\sqrt{5}\right) - \frac{1}{2}\left(2+\sqrt{5}\right)^{-2}\right]$

Hence (18.5) becomes: $\frac{1}{8}\left[\frac{1}{2}\left(2+\sqrt{5}\right)^2 + 2\ln\left(2+\sqrt{5}\right) - \frac{1}{2}\left(2+\sqrt{5}\right)^{-2}\right]$

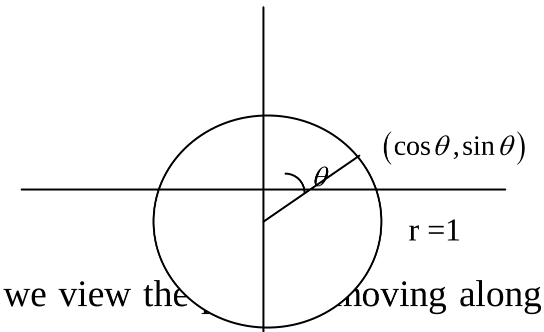
18.2 Length of Parameterized Curve

There is one other way in which we can describe a curve. Suppose that we look at a point which moves in the plane. Its coordinates can be given as a function of time t. Thus, we get two functions of t, say

$$x = f(t), \quad y = g(t),$$

We may view these as describing a point moving along a curve. The functions *f* and g give the coordinates of the point as function of t.

Example 18.3 Let, $x = r\cos\theta$, $y = r\sin\theta$. Then $(x, y) = \left(r\cos\theta, r\sin\theta\right)$ is a point on the circle.



As θ increases, we view the point moving along the circle in anticlockwise direction. The choice of letter θ really does not matter and we could use t instead. In particular, the angle θ is itself express as a function of time. For example, if a bug moves around the circle with uniform (constant) angular speed, then we can write $\theta = \omega\ t$, where ω is constant.

Then $x = \cos(\omega\, t)$, $y = \sin(\omega\, t)$.

When (x, y) is described by two function of t as above, we say that we have a parameterization of the curve in terms of parameter t.

This describes the motion of a bug around the circle with angular speed ω. Note that the parametric representation of a curve is not unique. For example $x = r\sin\theta$, $y = r\cos\theta$ also represents a point on the circle.

We shall now determine the length of a curve given by a parameterization. Suppose that our curve is given by

$$x = f(t),\ y = g(t), \text{ with } a \le t \le b$$

and assume that both f, g have continuous derivatives. With eqn (18.4) it is very reasonable to define the length of our curve (in parametric form) to be

$$l_a^b = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\ dt\,.$$

Observe that when a curve is given in usual form y = f(x) we can let

$$t = x = g(t) \quad \text{and } y = f(t).$$

This shows how to view the usual form as a special case of the parametric form. In that case $g'(t) = 1$ and the formula for the length in parametric form is seen to be the same as the formula we obtained before for a curve *y = f(x)* .It is also convenient to put the formula in the other standard notation for the derivative. We have

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = y'(t)$$

Hence the length of the curve can be written in the form

$$l_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Without loss of generality let

s(t) = length of the curve as function of t.

Thus we may write

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

This gives

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{f'(t)^2 + g'(t)^2}$$

Sometimes one writes symbolically

$$(ds)^2 = (dx)^2 + (dy)^2$$

To suggest the Pythagoras theorem i.e.,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Example 18.4 Find the length of the curve x = cos t , y = sin t between $t = 0, t = \pi$

Solution:

The length is the interval

$$\int_0^{\pi} \sqrt{(\sin t)^2 + (\cos t)^2}\; dt$$
$$= \pi \sqrt{}$$

If we integrate between 0 and 2π we would get 2π . This is the length of the circle of radius 1.

Example 18.5 Find the length of the curve x = e^t cos t , y = e^t sin t between t =1 and t = 2.

Solution:

$$l_1^2 = \int_1^2 \sqrt{\left[\left(e^t \cos t\right)'\right]^2 + \left[\left(e^t \sin t\right)'\right]^2}\; dt$$

$$= \int_1^2 \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2}\; dt$$

$$= \int_1^2 \sqrt{(e^{2t} \sin^2 t + e^{2t} \cos^2 t + e^{2t} \cos^2 t + e^{2t} \sin^2 t)}\; dt$$

$$= \sqrt{2}\int_{1}^{2} e^t dt = \sqrt{2}\left[e^2 - e\right]$$

Example 18.6 Find the length of the curve

$x = \cos^3\theta$, $y = \sin^3\theta$ for $0 \le \theta \le \pi/2$

Solution:

We have $\dfrac{dx}{d\theta} = 3\cos^2\theta(-\sin\theta)$

$$\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

Hence,

$$l_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} \sqrt{9\cos^4\theta + 9\sin^4\theta\cos^2\theta}\ d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta\sin^2\theta}\ d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta \quad \text{as } \sin\theta, \cos\theta > 0 \quad \text{for } 0 \le \theta \le \pi/2$$

Hence

$$l_0^{\frac{\pi}{2}} = 3\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\ d\theta = -\frac{3}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = \frac{3}{2}$$

<u>EXERCISES</u>

Find the length of the following curves:

1. $y = \ln x$, $\frac{1}{2} \leq x \leq 2$,

2. $y = 4 - x^2$, $-2 \leq x \leq 2$,

3. $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 1$ and $x = -1$

4. $y = \ln \cos x$, $0 \leq x \leq \frac{\pi}{3}$,

5. Find the length of the circle of radius r .

6. Find the length of the curve $x = \cos^3 t$, $y = \sin^3 t$ between t = 0 and t = π

7. Find the length of the curve x = 3t , y = 4t – 1 , $0 \leq t \leq 1$.

8. Find the length of the curve x = 1- cos t , y = t- sin t , $0 \leq t \leq 2\pi$.

9. Using exercise (9), find the length of the curve $r = \sin^2 \frac{\theta}{2}$ from 0 *to* π .

Ans.: 1. $\frac{\sqrt{5}}{2} + \ln\left(\frac{4+2\sqrt{5}}{1+\sqrt{5}}\right)$, 2. $2\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)^{\frac{1}{4}}$, 3. $e - \frac{1}{e}$, 4. $\ln\left(2+\sqrt{3}\right)$,

5. $2\pi r$, 6. 3, 7. 5, 8. 8 & 9. 2

Keywords: Rectification, length of curve, parametric form,

References

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.