# Lesson 17

# Tests for Convergence

## 17.1 Introduction

In this Lesson the convergence of Improper Integrals is studied.

Definition 16.1 if there exists a finite limit

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

Then this limit is called the value of the improper integral of the function f(x) on the interval $\left[a,+\infty\right]$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

In this case it is said that the improper integral exists or converges. If $\int_a^b f(x)\,dx$ as $b\to+\infty$ does not have a finite limit, one say that $\int_a^{+\infty} f(x)\,dx$ does not exist or diverges.

If $f(x)\geq 0$, the geometrical meaning of the improper integral can be seen as if the integral $\int_a^b f(x)\,dx$ expresses the area of region bounded by the curve y =

f(x), the x – axis and the ordinates x = a, x = b, it is natural to consider that the improper integral $\int_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded ( infinite ) region lying between the curve y = f(x), x = a and x-axis.

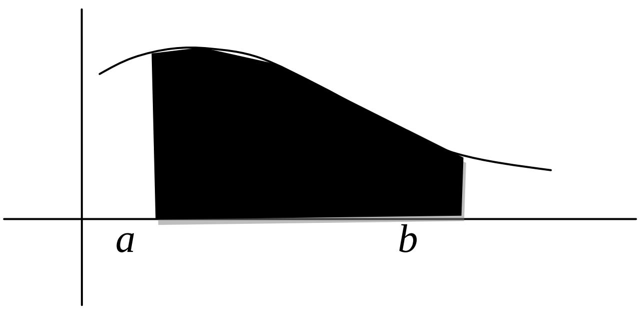
We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{\alpha \to -\infty} \int_{\alpha}^{a} f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx$$

The latter equation should be understood as if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

**Example 16.1** Find out at which p the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges and at which it diverges.




**Solution:**

Since (when $p \neq 1$)

$$\int_1^b \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p}\Big|_1^b = \frac{1}{1-p}\left(b^{1-p} - 1\right)$$

We have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \frac{1}{1-p}\left(b^{1-p} - 1\right)$$

Consequently, with respect to like this integral we conclude that if p > 1, then $\int_1^{+\infty} \frac{dx}{x^p} = \frac{1}{p-1}$ , and the integral converges.

If p < 1, then $\int_1^{+\infty} \frac{dx}{x^p} = \infty$ and integral diverges.

When p = 1, $\int_1^{+\infty} \frac{dx}{x^p} = \ln x\Big|_1^{+\infty} = \infty$, and the integral diverges.

***Note:*** We call the p-integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for p > 1, and diverges for $p \le 1$ which is in the comparison test of improper integral used.

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may useful in this respect.

**Theorem 17.1.** Let f and g be continuous function on the interval $[a,\infty)$ with $o \le f(x) \le g(x)\ \ \forall\ a \le x < \infty$.

If $\int_a^{+\infty} g(x)\,dx$ converges then $\int_a^{+\infty} f(x)\,dx$ also converges, and $\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

**Example 17.2** Investigate the integral $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ for convergence.

**Solution:**

It will be noted that when $1 \le x$

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$$

And $\int_1^{+\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{+\infty} = 1$

Consequently, $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ converges, and its value is less than 1. Hence $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ converges.

**Theorem 17.2.** If for all $x(x \ge a), 0 \le g(x) \le f(x)$ holds true and $\int_a^{+\infty} g(x)\,dx$ diverges, then the integral $\int_a^{+\infty} f(x)\,dx$ also diverges.

**Example 17.3** Find out whether the following integral converges or diverges.

$$\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}\,dx$$

**Solution:**

We note that $\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$

But $\int_1^{+\infty} \frac{dx}{x^{\frac{1}{2}}} = \infty$ as $p = \frac{1}{2} < 2$. Hence the given integral is divergent.

In the above two theorems we considered improper integrals of nonnegative functions. For the case of a function f(x) which changes its sign over an infinite interval we have the following result.

**Theorem17.3.** If the integral $\int_a^{+\infty} |f(x)|dx$ converges, then the integral $\int_a^{+\infty} f(x)dx$ also converges.

In this case, the later integral is called an absolutely convergent integral.

**Definition 17.1:** An integral $\int_a^{+\infty} f(x)dx$ converges conditionally if and only if $\int_a^{+\infty} f(x)dx$ converges but $\int_a^{+\infty} |f(x)|dx$ is not convergent.

**Example 17.3** Investigate the convergence of the integral $\int_1^{+\infty} \frac{\sin x}{x^3} dx$.

**Solution:**

Here, $\left|\frac{\sin x}{x^3}\right| \leq \left|\frac{1}{x^3}\right|$. But $\int_1^{+\infty} \frac{1}{x^3} dx$ convergent as p = 3.

Therefore, the integral $\int_1^{\infty} \frac{\sin x}{x^3} dx$ also converges.

## 17.2 The Integral of a Discontinuous Function

A function f(x) is defined and continuous when $a \le x < c$, and either not defined or discontinuous when x = c. In this case, one cannot speak of the integral $\int_a^c f(x)\,dx$ as limit of integral sums, because f(x) is not continuous on [a, c] and for this reason the limit may not exist.

The integral $\int_a^c f(x)\,dx$ of the function f(x) discontinuous at a point c is defined as follows:

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx$$

If the limit on the right exists, the integral is called an improper convergent integral, otherwise it is divergent. If the function f(x) is discontinuous at x = a of the interval [a, c] then by definition,

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{c} f(x)\,dx$$

If the function f*(x)* is discontinuous at some point x = $x_0$ inside the integral [a, c] , we put

$$\int_a^c f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^{c} f(x)\,dx$$

If both the improper integrals on the right hand side of the equation exist.

**Example 17.4** Test the convergence of the integral $\int_{-1}^{1} \frac{dx}{x^2}$.

**Solution:**

Since inside the interval of integration there exist a point x = 0, at which the integrand is not continuous, we express the integration as:

$$\int_{-1}^{1}\frac{dx}{x^2}=\lim_{\varepsilon\to 0}\int_{-1}^{0-\varepsilon}\frac{dx}{x^2}+\lim_{\varepsilon\to 0}\int_{0+\varepsilon}^{1}\frac{dx}{x^2}$$

$$=\lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x^2}+\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{dx}{x^2}$$

$$=\lim_{\varepsilon\to 0}-\frac{1}{x}\bigg|_{-1}^{-\varepsilon}-\lim_{\varepsilon\to 0}\frac{1}{x}\bigg|_{\varepsilon}^{1}$$

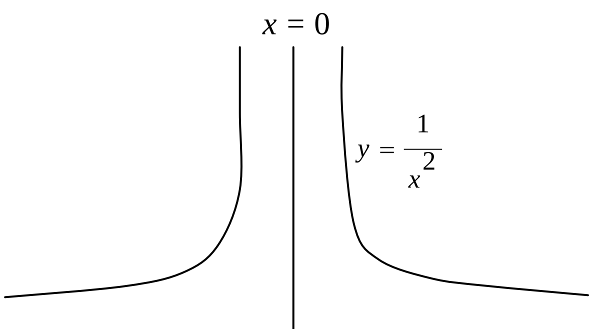
$$=-\lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}+1\right)-\lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon}\right)$$

But $-\lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}+1\right)=\infty$ and $-\lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon}\right)=\infty$ i.e., the integral diverges on [-1, 0] as well as on [0,1] .

Hence the given integral diverges on the entire interval [-1, 1].

It should be noted that if we had evaluated the given integral without paying attention to the discontinuity of the integrand at point x = 0, the result would have been wrong as $\int_{-1}^{1}\frac{dx}{x^2}=\frac{-1}{x}\bigg|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2$

This is impossible (Fig. 3)



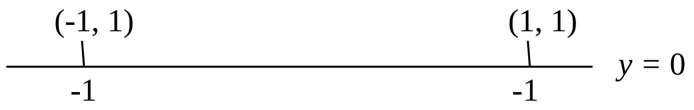



**Fig. 3**

***Note:*** If the function f(x), defined on the interval [a, b], and has finite number of discontinuity points $a_1, a_2, \ldots, a_n$ within the interval,

Then $\int_a^b f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \ldots.. + \int_{a_n}^{b} f(x)\,dx$

If each of the improper integrals on the right side of the equation converges then $\int_a^b f(x)\,dx$ is called convergent but if even one of these integrals diverges, then $\int_a^b f(x)\,dx$ too is called divergent.

For determining the convergence of improper integrals of discontinuous functions and for estimating their values, one can frequently make use of theorems similar to those used to estimate integrals within infinite limits.

**Theorem 17.3.** Let f(x) and g(x) be continuous functions in [a,c] except at x = c and at all points of this interval the inequalities $g(x) \geq f(x)$ hold and $\int_a^c g(x)\,dx$ converges, then $\int_a^c f(x)\,dx$ also converges.

**Theorem 17.4.** Let f(x) and g(x) be continuous functions on [a,c] except at x = c and at all points of this interval the inequalities f(x) $\geq$ g(x) $\geq 0$ hold and $\int_a^c g(x)\,dx$ diverges, then $\int_a^c f(x)\,dx$ also diverges.

**Theorem 17.5.** Let f*(x)* be a continuous function on [a, c] except at $x = c$ , and the improper integral $\int_a^c |f(x)|\,dx$ of the absolute value of this function converges, then the integral $\int_a^c f(x)\,dx$ of function of itself also converges. We frequently come across the improper integral of the following types.

$$\int_a^c \frac{dx}{(c-x)^p} \text{ , } also \int_a^c \frac{dx}{(x-a)^p}$$

It is easy to verify that $\int_a^c \frac{dx}{(c-x)^p}$ converges for p < 1 and diverges for p $\geq$ 1. Same applies also to 2nd.

**Example17.5** Does the integral $\int_0^1 \frac{dx}{\sqrt{x+4x^3}}$ converge?

**Solution:**

The integrand is discontinuous at x = 0.

Now $\frac{1}{\sqrt{x+4x^3}} \leq \frac{1}{\sqrt{x}}$

The improper integral $\int_0^1 \frac{dx}{x^{\frac{1}{2}}}$ $as\ \frac{1}{2} < 1$ exists and hence $\int_0^1 \frac{dx}{\sqrt{x+4x^3}}$ also exists.

## EXERCISES

Test the convergence of the following improper integrals:

1. $\int_0^\infty x \sin x \, dx$

2. $\int_1^\infty \frac{dx}{\sqrt{x}}$

3. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$

4. $\int_0^1 \frac{dx}{x^{\frac{1}{3}}}$

5. $\int_0^2 \frac{dx}{x^3}$

6. Let b > 2. Find the area under the curve $y = e^{-2x}$ between 2 and b. Does this area approach a limit when $b \to \infty$. If so what limit?

7. Can an improper integral $\int_a^\infty f(x) \, dx$ ever be transformed onto a proper integral by a change of variable?

**Ans.:** 1. The integral diverges, 2. The integral diverges, 3. $\pi$, 4. $\frac{3}{2}$, 5. The integral diverges, 6. $-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-4}$ , *yes* $\frac{1}{2}e^{-4}$ & 7. Yes, $f(x) = \frac{1}{x^2}$ , $x = \frac{1}{t}$.

**Keywords:** Convergence, absolutely convergence, comparison test.

**References**

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Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

## Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.