Lesson 16

Improper Integral

16.1 Introduction

Integral with infinite limits. Let a function f(x) be defined, positive and continuous for all values of x such that $a \le x < \infty$. Consider the integral

$$I(b) = \int_a^b f(x)\,dx$$

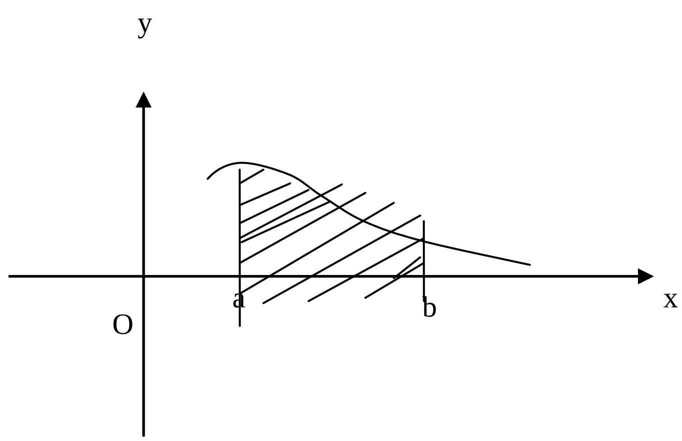

Fig. 1

This integral is meaningful for $b > a$. This integral varies with b and is continuous function of b. Let us consider the behavior of this integral when $b \to +\infty$ (Fig. 1). Definition 16.1 if there exists a finite limit

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

Then this limit is called the improper integral of the function f(x) on the interval $\left[a,+\infty\right]$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

In this case it is said that the improper integral exists or converges. If $\int_a^b f(x)\,dx$ as $b \to +\infty$ does not have a finite limit, one say that $\int_a^{+\infty} f(x)\,dx$ does not exist or diverges.

If $f(x) \geq 0$, the geometrical meaning of the improper integral can be seen as if the integral $\int_a^b f(x)\,dx$ expresses the area of region bounded by the curve y = f(x), the x – axis and the ordinates x = a, x = b, it is natural to consider that the improper integral $\int_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded (infinite) region lying between the curve *y = f(x), x = a* and *x*-axis.

We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^a f(x)\,dx = \lim_{\alpha\to-\infty}\int_\alpha^a f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx$$

The latter equation should be understood as if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

Example 16.1: Evaluate the integral $\int_{0}^{+\infty} \frac{dx}{1+x^2}$

Solution:

By the definition of improper integral we find

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big|_{0}^{b} = \frac{\pi}{2}$$

Note that this integral expresses the area of an infinite curvilinear trapezoid crosses x –axis as $x \to \infty$.

Example 16.2: Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Solution:

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}$$

The 2nd integral is equal to $\frac{\pi}{2}$ (see example 1)

Compute the First Integral:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_{b}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \tan^{-1} x \Big|_{b}^{0} = \lim_{b\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} b\right) = \frac{\pi}{2}$$

Hence, $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may useful in this respect.

Theorem 16.1: Let f and g be continuous function on the interval $[a,\infty)$ with $o \le f(x) \le g(x) \ \forall \ a \le x < \infty$.

If $\int_{a}^{+\infty} g(x)\,dx$ converges then $\int_{a}^{+\infty} f(x)\,dx$ also converges, and

$$\int_{a}^{+\infty} f(x)\,dx \le \int_{a}^{+\infty} g(x)\,dx$$

Theorem 16.1: The integral of a discontinuous function:

The integral $\int_{a}^{c} f(x)\,dx$ of the function f(x) discontinuous at a point c is defined as follows:

$$\int_a^c f(x)\,dx = \lim_{\varepsilon\to 0+}\int_a^{c-\varepsilon} f(x)\,dx$$

If the limit on the right exists, the integral is called an improper convergent integral, otherwise it is divergent. If the function f(x) is discontinuous at x = a of the interval [a,c] then by definition ,

$$\int_a^c f(x)\,dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{c} f(x)\,dx$$

If the function *f(x)* is discontinuous at some point x = x_0 inside the interval [a,c] , we put

$$\int_a^c f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^{c} f(x)\,dx$$

If both the improper integrals on the right hand side of the equation exist.

Example 16.3 Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$

Solution: $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}dx$

$$= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} (1-x)^{-\frac{1}{2}}\,dx$$

$$= -\left.\frac{(1-x)^{\frac{1}{2}}}{-\frac{1}{2}+1}\right|_0^{1-\varepsilon}$$

$$= -2\sqrt{1-x}\Big|_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} -2\left(\sqrt{\varepsilon} - 1\right) \; = 2$$

Example 16.4: Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2}$.

Solution:

Since inside the interval of integration there exist a point x = 0, at which the integrand is not continuous, we express the integration as:

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{0-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} -\frac{1}{x}\Big|_{-1}^{-\varepsilon} - \lim_{\varepsilon \to 0} \frac{1}{x}\Big|_{\varepsilon}^{1}$$

$$= -\lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} + 1 \right) - \lim_{\varepsilon \to 0} \left(1 - \frac{1}{\varepsilon} \right)$$

But $-\lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} + 1 \right) = \infty$ and $-\lim_{\varepsilon \to 0} \left(1 - \frac{1}{\varepsilon} \right) = \infty$ i.e., the integral diverges on [-1,0] as well as on [0,1] .

Hence the given integral diverges on the entire interval [-1, 1].

It should be noted that if we had evaluated the given integral without paying attention to the discontinuity of the integrand at point x = 0, the result would have been wrong as $\int_{-1}^{1}\frac{dx}{x^2}=\left.\frac{-1}{x}\right|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2$

For determining the convergence of improper integrals of discontinuous functions and for estimating their values, one can refer Lesson 17. These integrals are discussed in details in Lesson 17 also.

$$\int_a^c \frac{dx}{(c-x)^p}\text{ , also }\int_a^c \frac{dx}{(x-a)^p}$$

It is easy to verify that $\int_a^c \frac{dx}{(c-x)^p}$ converges for $p < 1$ and diverges for $p \geq 1$. Same applies also to 2nd integral.

EXERCISES

Evaluate the following improper integrals:

1. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

2. $\int_0^\infty e^{-x}dx$

3. $\int_0^\infty \frac{dx}{a^2+x^2}$

4. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

5. $\int_0^1 \ln x\, dx$

Ans.: 1. 1, 2. 1, 3. $\frac{\pi}{2a}\ (a>0)$, 4. $\frac{\pi}{2}$ & 5. 1

Keywords: Improper Integrals, Positive Function, Area Of the Region.

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Suggested Readings

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