Lesson 14

Asymptotes

14.1 Introduction

A straight line d is called an asymptote to a curve C (fig.1), if the distance δ distance from a point P of C to d approaches to zero as P recedes to infinity. Roughly speaking, a straight line is said to be an asymptote of a curve if it comes arbitrary close to that curve (but never touches the curve).

14.1.1 Asymptotes of Functions: If the graph of a function has an asymptote d, then we say that the function has an asymptote d. A function can have more than one asymptote. If an asymptote is parallel with the y-axis, we call it a vertical asymptote. If an asymptote is parallel with the x-axis,

we call it a horizontal asymptote. All other asymptotes are oblique asymptotes.

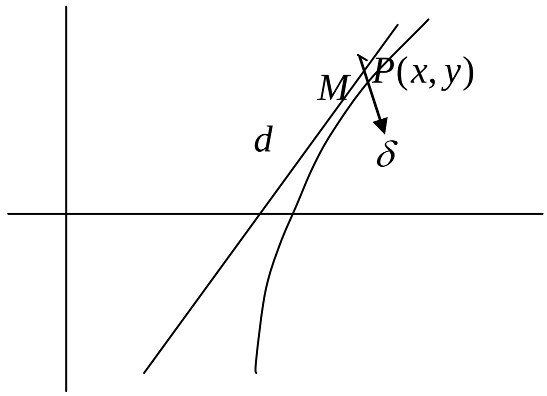

Fig. 1

Vertical Asymptotes

A straight line $x = a$ is a vertical asymptote to the the curve $y = f(x)$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$. Consequently, to find vertical asymptotes one has to find values of $x = a$ such that when they are approached by the function $y = f(x)$, the latter approaches infinity. Then the straight line is a vertical asymptote.

Example 14.1: The curve $y = \frac{2}{x-5}$ has a vertical asymptote $x = 5$, since $y \to \infty$ as $x \to 5^+$.

Example 14.2: The curve $y = \tan x$ has infinite number of vertical asymptotes at $x = \frac{n\pi}{2}$ for $n = 1,3,5,\cdots$, as $\tan x \to \infty$ when $x \to \pm\frac{n\pi}{2}$.

Example 14.3: The curve $y = \frac{x^2+3x+2}{x+2}$ has no vertical asymptote at $x = -2$ as $\lim_{x\to-2}\frac{x^2+3x+2}{x+2} = -1$.

14.2 Horizontal Asymptotes

A line $y = b$ is a horizontal asymptote of a function $f(x)$ iff $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, with $b \in R$.

Examples 14.4: The curve $y = \frac{3x^2-4x-1}{6x^2-6}$ has horizontal asymptote as

$\lim_{x\to\infty}\frac{3x^2-4x-1}{6x^2-6} = \frac{1}{2}$. So, $y = \frac{1}{2}$ is a horizontal asymptote of the function $\frac{3x^2-4x-1}{6x^2-6}$.

14.3 Oblique Asymptotes/Inclined Asymptotes

Let the curve $y = f(x)$ have an inclined or oblique asymptote d(fig.1) whose equation is $y = mx + c$.

Here m and c are unknown real numbers to be determined. Let $PM = \delta$ be the perpendicular distance of any point $P(x, y)$ on the curve to the line $y = mx + c$.

Hence, $\delta = \frac{y-mx-c}{\sqrt{1+m^2}}$. Now $\delta \to 0$ as $x \to \infty$. Hence, $\lim_{x\to\infty}[y - mx - c] = 0$. i.e., $\lim_{x\to\infty}[y - mx] = c$, hence

$$\lim_{x\to\infty}[\frac{y}{x} - m] = \lim_{x\to\infty}[y - mx].\lim_{x\to\infty}\frac{1}{x}$$

$= c.0 = 0.$

So $m = \lim_{x\to\infty}\frac{y}{x}$.

Example 14.5: Find the asymptotes to the curve $y = \frac{x^2+2x-1}{x}$

Solution:

When $x \to 0^-$, $y \to +\infty$, and $x \to 0^+$, $y \to -\infty$, hence the straight line $x = 0$ is a vertical asymptote of the above curve.

Next to find the asymptotes of the form $y = mx + c$, i.e., the inclined asymptote.

$$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{x^2+2x-1}{x^2}$$

$$= \lim_{x\to\infty} [1 + \frac{2}{x} - \frac{1}{x^2}] = 1$$

$$c = \lim_{x\to\infty} [y - mx] = \lim_{x\to\infty} [y - x]$$

$$= [\frac{x^2+2x-1}{x} - x] = \lim_{x\to\infty} [2 - \frac{1}{x}] = 2.$$

Hence $y = x + 2$ is an inclined asymptotes to the given curve.

Example 14.6: Find the oblique asymptotes to the curve $y = \sqrt{x^2 - 1} + 2$

Solution:

$y = -x + 2$

14.3.1 Tutorial Discussion

- An asymptote is a straight line which acts as a boundary for the graph of a function.

- When a function has an asymptote (and not all functions have them) the function gets closer and closer to the asymptote as the input value to the function approaches either a specific value a or positive or negative infinity.
- The functions most likely to have asymptotes are rational functions
- Vertical asymptotes occur when the following condition is met:

The denominator of the simplified rational function is equal to 0.

Remember, the simplified rational function has cancelled any factors common to both the numerator and denominator.

e.g., Given the function $f(x) = \frac{2-5x}{2+2x}$

The first step is to cancel any factors common to both numerator and denominator. In this case there are none.

The second step is to see where the denominator of the simplified function equals 0. $2 + 2x = 0$ implies $x = -1$.

The vertical line $x = -1$ is the only vertical asymptote for the function. As the input value x to this function gets closer and closer to -1 the function itself looks and acts more and more like the vertical line $x = -1$.

Example 14.7 $f(x) = \frac{2x^2+10x+12}{x^2-9}$

First simplify the function. Factor both numerator and denominator and cancel any common factors.

$$f(x) = \frac{2x^2+10x+12}{x^2-9} = \frac{(x+3)(2x+4)}{(x+3)(x-3)} = \frac{2x+4}{x-3}$$

The asymptote(s) occur where the simplified denominator equals 0. i.e., $x - 3 = 0$.

The vertical line $x = 3$ is the only vertical asymptote for this function. As the input value x to this function gets closer and closer to 3 the function itself looks more and more like the vertical line $x = 3$.

Example 14.8 If $g(x) = \frac{x-5}{x^2-x-6}$

Factor both the numerator and denominator and cancel any common factors.
In this case there are no common factors to cancel.

$$\frac{x-5}{x^2-x-6} = \frac{x-5}{(x+2)(x-3)}$$

The denominator equals zero whenever either $x+2=0$ or $x-3=0$. Hence this function has two vertical asymptotes, one at $x=-2$ and the other at $x=3$.

5. Horizontal Asymptotes

Horizontal asymptotes occur when either one of the following conditions is met (you should notice that both conditions cannot be true for the same function).

- The degree of the numerator is less than the degree of the denominator. In this case the asymptote is the horizontal line $y=0$.
- The degree of the numerator is equal to the degree of the denominator. In this case the asymptote is the horizontal line $y=\frac{a}{b}$ where a is the leading coefficient in the numerator and b is the leading coefficient in the denominator.

When the degree of the numerator is greater than the degree of the denominator there is no horizontal asymptote.

Example 14.9 $f(x)=\frac{x^2-3x+5}{x^3-27}$

then there is a horizontal asymptote at the line $y=0$ because the degree of the numerator 2 is less than the degree of the denominator 3.

This means that as x gets larger and larger in both the positive and negative directions $(x \to \infty)$ and $(x \to -\infty)$ the function itself looks more and more like the horizontal line $y = 0$

Find the vertical asymptotes, horizontal asymptotes and inclined asymptotes for each of the following functions Problems:

Exercises:

Find the asymptotes of the following curves:

1. $f(x) = \frac{x^2+2x-15}{x^2+7x+10}$

Solution: Vertical: $x = -2$ Horizontal: $y = 1$ Inclined: none

2. $g(x) = \frac{2x^2-5x+7}{x-3}$

Solution: Vertical: $x = 3$ Horizontal: none Inclined: $y = 2x + 1$

3. $y = \frac{x^2+1}{1+x}$ Ans. $x = -1, y = x - 1$

4. $y = x + e^{-x}$ Ans. $y = x$

5. $y = a^3 - x^2$ Ans. No asymptotes

6. $y = x\ln(e + \frac{1}{x})$ Ans. $x = -\frac{1}{e}$, $y = x + \frac{1}{e}$

7. $y = x\,e^{\frac{1}{x^2}}$ Ans. $x = 0$

8. Sketch the function $y = \frac{x^2-3}{2x-4}$

Keywords: Asymptotes, horizantal, vertical and inclied asymptotes.

References

W. Thomas, Finny (1998). Calculus and Analytic Geometry, 6th Edition, Publishers, Narsa, India.

R. K. Jain, and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol, (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.