Lesson 13

Curvature

13.1 Introduction

Curvature measures the extent to which a curve is not contained in a straight line. It curvature measures how curved the curve is. We have heard the comparison of bending or curvature of a road at two of its points. The curvature of a straight line is zero. It also measures how fast the tangent vector turns as a point moves along the curve.

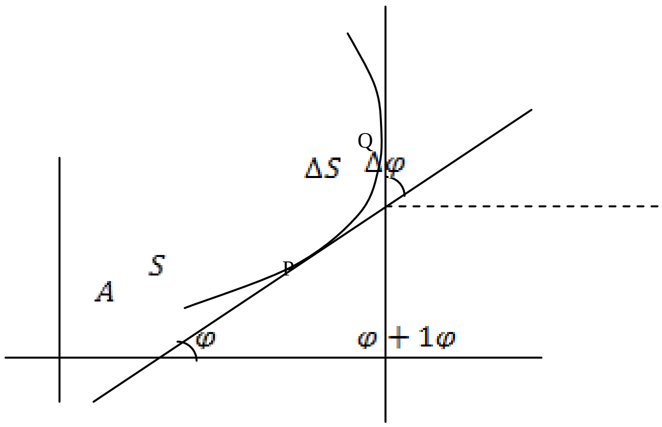

Fig.1.

Let A be a fixed point on the curve. Let arc $AP = s$, and arc $AQ = s + \Delta s$, so that arc $PQ = \Delta s$. Let ϕ, $\phi + \Delta\phi$ be the angles which the tangents at P and Q make with some fixed line (say x- axis). $\Delta\phi$ denotes the angle formed by these tangents. The symbol $\Delta\phi$ also denotes the angle through which the tangent turns from P and Q through a distance Δs. $\Delta\phi$ will be large or small, as compared with Δs, depending the degree of the sharpness of the bend. This suggests the following definitions:

The curvature of the curve at P is defined as $\lim_{Q\to P}|\frac{\Delta\phi}{\Delta s}| = |\frac{d\phi}{ds}|$.

The reciprocal of curvature $\rho = \frac{ds}{d\phi}$ is the radius of curvature.

Length of Arc as a Function, Derivative of Arc.

Let $y = f(x)$ be the equation of a given curve on which we take a fixed point A. Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be the variable points on the curve with arc $AP = s$ and arc $AQ = s + \Delta s$ so that arc $PQ = \Delta s$.

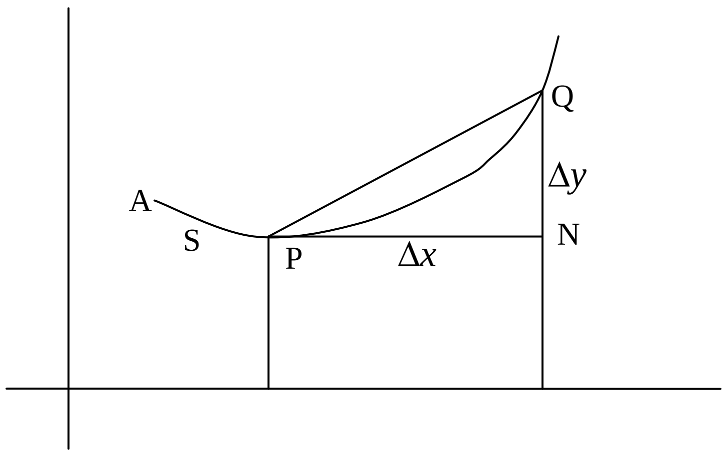

Fig. 2.

$$chordPQ^2 = PN^2 + NQ^2 = \Delta x^2 + \Delta y^2$$

$\Rightarrow$

$$\left(\frac{chord\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$\Rightarrow$

$$\left[\frac{chord\ PQ}{arc\ PQ}\right]^2\left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$\lim_{Q\to P}\frac{chord\ PQ}{arc\ PQ} = 1$, taking limit $\lim_{Q\to P}$ both sides we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$\Rightarrow$

$$\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$$

Radius of Curvature: Cartesian Equations

We define the absolute value of $\frac{d\phi}{ds}$ as the curvature and denote it by $\kappa = |\frac{d\phi}{ds}|$.

Consider the curve $y = f(x)$, we note that $\tan\phi = \frac{dy}{dx}$ and, therefore,

$$\phi = \tan^{-1}(\frac{dy}{dx})$$

Differentiating this with respect to x, we have

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2}$$

As $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$, we have

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2}}{\sqrt{1+(\frac{dy}{dx})^2}} = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}$$

Hence $\rho = |\frac{ds}{d\phi}| = \frac{(1+{y_1}^2)^{\frac{3}{2}}}{y_2}$, where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

Note: If $\rho = \frac{ds}{d\phi}$, the radius of curvature, ρ, is positive or negative according as $\frac{d^2y}{dx^2}$ is +ve or -ve i.e., accordingly as the curve is convex downward or convex upward. But we consider ρ is +ve here. Curvature is zero at point of inflection. Since ρ is independent of the choice of x-axis and y-axis, interchanging x and y, we see that ρ, is given by

$$\frac{[1+(\frac{dx}{dy})^2]^{\frac{3}{2}}}{|\frac{d^2x}{dy^2}|}.$$

Curvature- parametric Equation

Given $x = f(t)$, $y = F(t)$. $f'(t) \neq 0$.

$$\frac{dy}{dx} = \frac{F'(t)}{f'(t)}, \frac{d^2y}{dx^2} = \frac{f'F'' - F'f''}{[f']^3}$$

Hence the curvature $\kappa = \frac{|f'F'' - F'f''|}{[(f')^2 + (F')^2]^{\frac{3}{2}}}$ $\kappa = \frac{ds}{d\phi}$

Curvature- polar Equation

Let $r = f(\theta)$ be the given curve in polar co-ordinates. Now its cartesian coordinates are of the form $x = r\cos\theta$, $y = r\sin\theta$. i.e., $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. Now

$$\frac{dx}{d\theta} = \frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{d^2x}{d\theta^2} = \frac{d^2r}{d\theta^2}\cos\theta - 2\frac{dr}{d\theta}\sin\theta - r\cos\theta$$

$$\frac{d^2y}{d\theta^2} = \frac{d^2r}{d\theta^2}\sin\theta + 2\frac{dr}{d\theta}\cos\theta - r\sin\theta$$

substituting the latter expressions in the previous parametric-form, we have

$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{\frac{3}{2}}} (*)$$

We know

$$\kappa = \frac{|f'F'' - F'f''|}{[(f')^2 + (F')^2]^{\frac{3}{2}}}$$

numerator becomes

$$(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \times (r'\cos\theta - r\sin\theta)$$

$$-(r'\sin\theta + r\cos\theta) \times (r''\cos\theta - 2r'\sin\theta - r\cos\theta)$$

$$= r''r'\sin\theta\cos\theta + 2r'^2\cos^2\theta - rr'\sin\theta\cos\theta$$

$$-rr''\sin^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta$$

$$-r'r''\sin\theta\cos\theta + 2r'^2\sin^2\theta + rr'\sin\theta\cos\theta$$

$$-rr''\cos^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= r^2(sin^2\theta + \cos^2\theta) + 2r'^2(\cos^2\theta + \sin^2\theta)$$

$$-rr''(sin^2\theta + \cos^2\theta)$$

To check we can observe that

$$[f'\mathrm{F}'' - \mathrm{F}'f''] = |r^2 + 2r'^2 - rr''|$$

denominator becomes

$$(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$= r'^2\cos^2\theta + r^2\sin^2\theta - 2rr'\sin\theta\cos\theta$$

$$r'^2\sin^2\theta + r^2\cos^2\theta + 2rr'\sin\theta\cos\theta$$

$$= r'^2(\cos^2\theta + \sin^2\theta) + r^2(sin^2\theta + \cos^2\theta)$$

$$[(f')^2 + (\mathrm{F}')^2]^{\frac{3}{2}} = \left(r'^2 + r^2\right)^{\frac{3}{2}}$$

Hence

$$\kappa = \frac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{\frac{3}{2}}}$$

The radius of curvature is

$$\rho = \frac{(r^2+r'^2)^{\frac{3}{2}}}{|r^2+2r'^2-rr''|}$$

Example 1: Determine the radius of curvature of the curve $r = a\theta\ (a > 0)$

Solution:

$\frac{dr}{d\theta} = a, \frac{d^2r}{d\theta^2} = 0$

Hence

$$\rho = \frac{(a^2\theta^2+a^2)^{\frac{3}{2}}}{a^2\theta^2+2a^2} = \frac{a(\theta^2+1)^{\frac{3}{2}}}{\theta^2+2}$$

* We know

$$\kappa = \frac{|f'F''-F'f''|}{[(f')^2+(F')^2]^{\frac{3}{2}}}$$

numerator becomes

$$(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \times (r'\cos\theta - r\sin\theta)$$

$$-(r'\sin\theta + r\cos\theta) \times (r''\cos\theta - 2r'\sin\theta - r\cos\theta)$$

$$= r''r'\sin\theta\cos\theta + 2r'^2\cos^2\theta - rr'\sin\theta\cos\theta$$

$$-rr''\sin^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta$$

$$-r'r''\sin\theta\cos\theta + 2r'^2\sin^2\theta + rr'\sin\theta\cos\theta$$

$$-rr''\cos^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= r^2(sin^2\theta + \cos^2\theta) + 2r'^2(\cos^2\theta + \sin^2\theta)$$

$$-rr''(sin^2\theta + \cos^2\theta)$$

$$= |r^2 + 2r'^2 - rr''|$$

denominator becomes

$$(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$= r'^2\cos^2\theta + r^2\sin^2\theta - 2rr'\sin\theta\cos\theta$$

$$r'^2\sin^2\theta + r^2\cos^2\theta + 2rr'\sin\theta\cos\theta$$

$$= r'^2(\cos^2\theta + \sin^2\theta) + r^2(sin^2\theta + \cos^2\theta)$$

$$r'^2 + r^2$$

Hence

$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{\frac{3}{2}}} (*)$$

Example 2: Find the radius of curvature of $r = a\sec^2\frac{\theta}{2}$

Ans.: $\rho = 2a\sec^3\frac{\theta}{2}$

Example : Find the radius of curvature of $x = 3t^2$, $y = 3t - t^3$ for $t = 1$,

Ans.: $\rho = 6$

Example : Find the curvature of the hyperbola $xy = 1$ at $(1,1)$.

Solution:

$y' = -\frac{1}{x^2}$ and $y'' = \frac{2}{x^3}$. So

$$\kappa = \frac{\frac{2}{x^3}}{[1+(\frac{1}{x^4})]^{\frac{3}{2}}} = \frac{2}{x^3}\frac{x^6}{(x^4+1)^{\frac{3}{2}}} = \frac{2x^3}{(x^4+1)^{\frac{3}{2}}}$$

When $x = 1$, $\kappa = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Example 3: For what value of x is the radius of curvature of $y = e^x$ smallest?

Solution:

$y' = y'' = e^x$, $\kappa = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ and radius of curvature ρ is $\frac{(1+e^{2x})^{\frac{3}{2}}}{e^x}$. Then

$$\frac{d\rho}{dx} = \frac{e^x.\frac{3}{2}(1+e^{2x})^{\frac{1}{2}}(2e^{2x}) - e^x(1+e^{2x})^{\frac{3}{2}}}{e^{2x}}$$

$$= \frac{(1+e^{2x})^{\frac{1}{2}}[3e^{2x}-(1+e^{2x})]}{e^x}$$

$$= \frac{(1+e^{2x})^{\frac{1}{2}}(2e^{2x}-1)}{e^x}$$

etting $\frac{d\rho}{dx} = 0$, we find $2e^{2x} = 1$, $2x = \ln\frac{1}{2} = -\ln 2$, $x = -\frac{(\ln 2)}{2}$. As the second derivative at this point is positive, $x = -\frac{(\ln 2)}{2}$ is the point which gives the smallest radius of curvature.

Example 4: Find the radius of curvature at any point on the curves: $y = c\cosh\frac{x}{c}$

Solution:

$y' = c\sinh\frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c}$, $y'' = \frac{1}{c}\cosh\frac{x}{c}$

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{|\frac{d^2y}{dx^2}|} = \frac{[1+\sinh^2\frac{x}{c}]^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{(\cosh^2\frac{x}{c})^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c}$$

$$y^2 = c^2\cosh^2\frac{x}{c}$$

implies

$$\frac{y^2}{c} = c\cosh^2\frac{x}{c}$$

i.e. $\rho = \frac{y^2}{c}$.

Example : Find the radius of curvature at the origin of the curve

$$y - x = x^2 + 2xy + y^2$$

Solution:

$$\frac{dy}{dx} - 1 = 2x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx}$$

$\Rightarrow \frac{dy}{dx}|_{(0,0)} = 1$

$$\frac{d^2y}{dx^2} = 2 + 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2(\frac{dy}{dx})^2 + 2y\frac{d^2y}{dx^2}$$

which implies $\frac{d^2y}{dx^2}|_{(0,0)} = 8$.

$$\rho = \frac{(1+y_1)^{\frac{3}{2}}}{y_2} = \frac{2^{\frac{3}{2}}}{8} = \frac{\sqrt{2}}{4}$$

Example 5: Find the curvature of the cycloid $x = a(t - \text{sin}t)$, $y = a(1 - \text{cos}t)$ at an arbitrary point (x, y).

Solution:

$\frac{dx}{dt} = a(1 - \cos t)$, $\frac{d^2x}{dt^2} = a\sin t$, $\frac{dy}{dt} = a\sin t$, $\frac{d^2y}{dt^2} = a\cos t$. Using this parametric formula $\kappa = \frac{|f'F'' - F'f''|}{[(f')^2 + (F')^2]^{\frac{3}{2}}}$, we obtain

$$\kappa = \frac{|a(1-\cos t)a\cos t - a\sin t . a\sin t|}{[a^2(1-\cos t)^2 + a^2\sin^2 t]^{\frac{3}{2}}}$$

$$= \frac{|a^2(\cos t - \cos^2 t - \sin^2 t)|}{[2a^2(1-\cos t)]^{\frac{3}{2}}}$$

$$= \frac{|\cos t - 1|}{2^{\frac{3}{2}}a(1-\cos t)^{\frac{3}{2}}}$$

$$= \frac{1}{2^{\frac{3}{2}}a(1-\cos t)^{\frac{1}{2}}} = \frac{1}{|4a\sin\frac{t}{2}|}$$

When $t = \pi$, $\kappa = \frac{1}{|4a|}$

Questions: Answer the following questions.

1. Find the curvature of the curve $b^2x^2 + a^2y^2 = a^2b^2$ at the point (a,b) and (a,0)

2. Find the curvature of the curve $16y^2 = 4x^4 - x^6$ at the point (2,0)

3. Find the curvatur e of the curve $xy = 12$ at the point (3,4)

Questions: Find the radius of curvature of the following curves at the indicated points.

4. $y = x^3$ at the point (4,8)

5.$x^{\ 2} = 4ay$ at the point $(0,0)$

6. $y = \ln x$ at the point (1,0)

7. $y = \sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$

8. Find the point of the curve $y = e^x$ at which the radius of curvature is minimum.

Ans.: 1. $\frac{b}{a^2}, \frac{a}{b^2}$, 2. $\frac{1}{2}$, 3. $\frac{24}{125}$, 4. $\frac{80\sqrt{10}}{3}$, 5. 29, 6. $2\sqrt{2}$, 7. 1 & 8. $-\frac{1}{2}ln2, \frac{\sqrt{2}}{2}$

References

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Jain, R. K. and Iyengar, SRK. (2010), Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.