Lesson 12

Convexity, Concavity and Points of Inflexion

12.1 Introduction

In the plane, we consider a curve $y = f(x)$, which is the graph of a single-valued differentiable function $f(x)$.

Definition 12.1: We say that the curve is convex downward bending up on the interval (b, c) if all points of the curve lie above the tangent at any point on the interval. Or when the curve turns anti-clock wise we call it is convex downward (concave upward) (see Fig. 1).

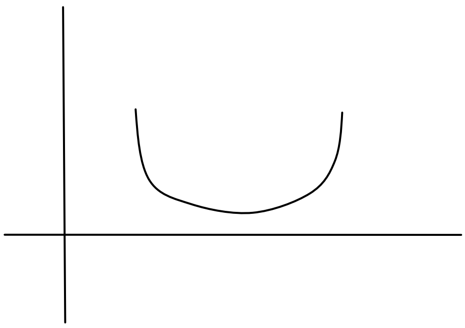


Fig.1. (Convex downward/Bending up)

Definition: We say that a curve is convex upwards for bending down on the interval (a, b) if all points of the curve lie below the tangent at any point on the interval. Or when the curve turns clock-wise we say it is convex upward (concave downward) (see Fig. 2).

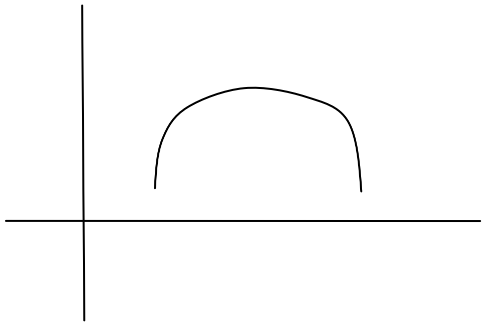


Fig. 2. (Convex upward / Bending down)

The curve has a point of inflexion at P, at which the curve changes from convex upwards to convex downwards and vice-versa.

Theorem 1: If for all points of an interval (a, b), $f''(x) < 0$, the curve $y = f(x)$ on this interval is convex upward. If $f''(x) > 0$, the curve is convex downward.

If $f''(x) < 0 \; \forall \, x \in (a, b) \Rightarrow y = f(x)$ is convex upward on (a, b).

If $f''(x) > 0 \; \forall \, x \in (a, b) \Rightarrow y = f(x)$ is convex doward on (a, b).

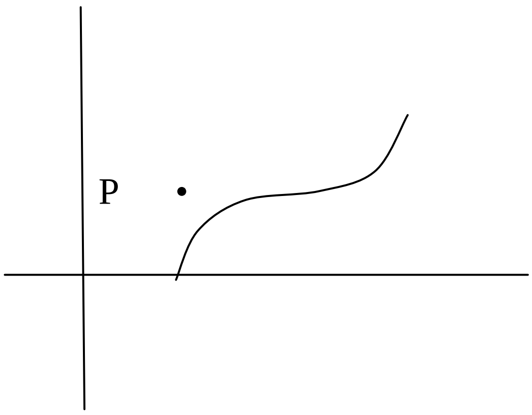
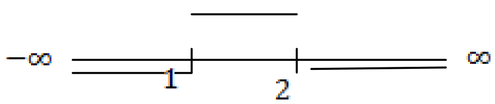


Fig. 3. (Inflexion point)

Example 12.1: Find the ranges of values of x for which the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is convex downwards, convex upwards, and also determine the point of inflection.

Solution:

$$y' = 4x^3 - 18x^2 + 24x + 5,$$



$$y'' = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$

Now on the interval $(-\infty, 1)$, $x - 1 < 0$, $x - 2 < 0$, hence $y'' > 0$. If $x > 2$, $x > 1$, i.e., $x - 2 > 0$ and $x - 1 > 0$. Hence for $x \in (2, \infty)$, $y'' > 0$. Now on the interval $(1,2)$, $y'' < 0$. Hence the curve is convex downward on the interval $(-\infty, 1)$ and $(2, \infty)$. Convex upwards on $(1,2)$. The curve has inflection points at $= 1$ and $x = 2$ as y'' changes sign. At $x = 1$, $y = 19$ and at $x = 2$, $y = 33$. i.e., $(1,19)$ and $(2,33)$ are two points of inflection of the curve.

Example 12.2: Determine the intervals where the graph of the function is convex downward and convex upward of $f(x) = \frac{x}{2x-1}$

Solution:

$$f(x) = [(x - \tfrac{1}{2}) + \tfrac{1}{2}]/[2(x - \tfrac{1}{2})] = \tfrac{1}{2}\{1 + [1/(2x - 1)]\}.$$

Hence,

$$f'(x) = \tfrac{1}{2}[-1/(2x - 1)^2].2 = -\frac{1}{(2x-1)^2}.$$

Then $f''(x) = 4/(2x - 1)^3$. For $x > \frac{1}{2}$, $2x - 1 > 0$, $f''(x) > 0$, the graph is convex downward. For $x < \frac{1}{2}$, $2x - 1 < 0$, $f''(x) < 0$, the graph is convex upward. There is no inflection point, since $f(x)$ is not defined when $x = 1/2$.

Example 12.3: Determine the intervals where the graph of the function is convex downward and convex upward of $f(x) = 5x^4 - x^5$,

Solution:

$f'(x) = 20x^3 - 5x^4$, and $f''(x) = 60x^2 - 20x^3 = 20x^2(3 - x)$. So, for $0 < x < 3$ and for $x < 0$, $3 - x > 0$, $f''(x) > 0$, the graph is convex downward. For $x > 3$, $3 - x < 0$, $f''(x) < 0$, and the graph is convex upward.

There is an inflection point at $(3,162)$. There is no inflection point at $x = 0$, the graph is convex downward for $x < 3$.

Example 12.4: Find the point of inflection of the curve $y = (\ln x)^3$,

Solution:

$y'(x) = 3(\ln x)^2.\frac{1}{x}$, $y'' = \frac{3\ln x}{x^2}(2 - \ln x)$. $y'' = 0$ if $\ln x = 0$, or $\ln x = 2$. i.e., $x = 1$ or $x = e^2$. Now y'' changes sign from negative to positive as x passes through 1 and changes sign from positive to negative as x passes through e^2. Thus $(1,0)$ and $(e^2,8)$ are two points of inflection of the given curve.

Example 12.5: What conditions must the coefficients a, b, c satisfy for the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection?

Solution:

$y'' = 12ax^2 + 6bx + 2c$ has a point of inflection iff the equation $2ax^2 + 6bx + 2c = 0$ has different real roots. i.e., discriminant $D = 9b^2 - 24ac > 0$ is positive. i.e. $3b^2 > 8ac$.

Questions: Answer the following questions.

1. Determine all the inflexion points of $sin\, x$.

2. Determine all the inflexion points of $cos\, x$.

3. Determine all the inflexion points of $f(x) = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

4. Sketch the curve$y = sin^2 x$. Determine the inflexion points. Compare with

   graph of $|\sin x|$.

5. Determine the inflexion points and the intervals of convex downward / bending up and convex upward / bending down for the following curve
6. $y = x + \frac{1}{x}$

7. $y = \frac{x}{x^2+1}$

8. $y = \frac{x}{x^2-1}$

9. Sketch the curve $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$

10. Pint of inflexion of $y = x^4$.

Keywords: Convex up, Convex down, Inflexion Point.

References

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Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

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Suggested Readings

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