Lesson 11

Lagrange's Multiplier Rule / Constrained Optimization

11.1 Introduction

We presents an introduction to optimization problems that involve finding a maximum or a minimum value of an objective function $f(x,y)$ subject to a

constraint of the form $g(x,y) = k$.

Maximum and Minimum. Finding optimum values of the function $f(x,y)$

without a constraint is a well known problem in calculus. One would normally use the gradient to find critical points (gradient (∇f) vanishes). Then check all

stationary and boundary points to find optimum values.

Example 1.$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

$f(x,y) = 2x^2 + y^2, f_x(x,y) = 4x = 0, f_y(x,y) = 2y = 0, f(x,y)$

has a critical/ stationary point at (0,0).

The Hessian: A common method of determining whether or not a function has an extreme value at a stationary point is to evaluate the hessian of the function of n variables at that point. where the hessian is defined as

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{pmatrix}$$

A square matrix of order $n \times n$ is said to be positive definite if its leading principal minors are all positive.

For n=2, we have

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Second Derivative Test: The Second derivative test determines the optimality of stationary point x according to the following rules:

Let $\frac{\partial^2 f}{\partial x^2} = A, \frac{\partial^2 f}{\partial x \partial y} = B, \frac{\partial^2 f}{\partial y^2} = C$, and $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at the point (x, y), then

1. If $A > 0$ and $AC - B^2 > 0$ at the point (x, y), then f has a local minimum at (x, y).
2. If $A < 0$ and $AC - B^2 > 0$ at the point (x, y), then f has a local maximum at (x, y).
3. If $AC - B^2 < 0$ at (x, y), then (x, y) is a saddle point of f.
4. If $AC - B^2 = 0$, further investigation is required.

In the above Example 1,

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

Therefore $f(x, y)$ has a minimum at (0,0) as $4 > 0$ and determinant of the matrix is $8 > 0$.

11.1.1 Constrained Maximum and Minimum

When finding the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$, the stationary points found above will not work. This new problem can be thought of as finding extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the surface $g(x, y) = k$. The value of $f(x, y)$ is maximized (minimized) when the surfaces touch each other,i.e , they have a common tangent for line.

This means that the surfaces, gradient vectors at that point are parallel, hence,

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The number λ in the equation is known as the Lagrange multiplier.

11.2 Lagrange multiplier method

The Lagrange multiplier methods solves the constrained optimization problem by transforming it into a non-constrained optimization problem of the form:

$$L(x, y, \lambda) = f(x, y) + \lambda(k - g(x, y))$$

or $(g - k))$. Then finding the gradient and Hessian as was done above will determine any optimum values of $L(x, y, \lambda)$.

Suppose we want to find optimum values for the following:

Example 11.2: $f(x, y) = 2x^2 + y^2$ subject to $x + y = 1$.

Then the Lagrangian method will result in a non-constrained function.

$L(x, y, \lambda) = 2x^2 + y^2 + \lambda(1 - x - y)$. The gradient for this new function is

$$\frac{\partial L}{\partial x} = 4x - \lambda = 0$$

$$\frac{\partial L}{\partial y} = 2y - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x - y = 0$$

Solving x, y, λ, we obtain $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $\lambda = \frac{4}{3}$.

The Hessian matrix at the stationary point

$$H(L) = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Since $H(L)$ is positive definite the solution $x = \frac{1}{3}$, $y = \frac{2}{3}$ minimizes

$f(x, y) = 2x^2 + y^2$ subject to $x + y = 1$ with $f(\frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$

Example 11.3: Find the rectangle of parameter *l* which has maximum area i.e., Maximize xy subject to

$$2(x+y)=l$$

Solution:

$$L(x,y,\lambda)=xy+\lambda 2(x+y)-l).$$

$$\frac{\partial L}{\partial x}=y+2\lambda=0$$

$$\frac{\partial L}{\partial y}=x+2\lambda=0$$

i.e., $x=y=-2\lambda$ i.e., $-8\lambda=l \Rightarrow \lambda=-\frac{l}{8}$.

$x=y=\frac{l}{4}$, so that the rectangule of maximum area is a square.

Example 11.4 Find the shortest distance from the point (1,0) to the parabola $y^2=4x$, i.e., Minimize $(x-1)^2+y^2$ subject to $y^2=4x$.

$$L(x,y,\lambda)=(x-1)^2+y^2+\lambda(y^2-4x)$$

$$\frac{\partial L}{\partial x}=2(x-1)-4\lambda=0, \frac{\partial L}{\partial y}=2y+2\lambda y=0$$

$$y^2-4x=0$$

Now $2y + 2\lambda y = 0 \Rightarrow y = 0 \; or \; \lambda = -1$

If $\lambda = -1,$ then $x = -1$, from $2(x-1) - 4\lambda = 0$

Hence $y = 0 \Rightarrow x = 0$

$x = -1$

Now $\mathrm{y}^2 = 4^{\mathrm{n}}, \mathrm{y}^2 = -4$, so $\mathrm{y} = \sqrt{-4}$ not possible no real value.

Hence $\mathrm{y} = 0, \mathrm{x} = 0,$

i.e., $\lambda = -\frac{1}{2}$

Hence the only solution is $x = 0, y = 0, \lambda = -\frac{1}{2}$ and the required distance is unity.

Questions: Answer the following question

1. Determine the maximum value of the n-th root of a product of numbers $x_1, x_2, \cdots, x_n$ provided that their sum is equal to a given number a. Thus the problem is stated as follows: it is required to find the maximum of the function $z = \sqrt[n]{x_1 . x_2 \cdots, x_n}$ subject to $\sum_{i=1}^{n} x_i - a = 0$, $x_i > 0$, for all i.

References

W. Thomas, Finny (1998). Calculus and Analytic Geometry, 6th Edition,

Publishers, Narsa, India.

Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3 rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.