

Lesson 10**Maximum and Minimum of function of two variables****10.1 Introduction**

We say that a function $z = f(x, y)$ has a maximum (local) at a point (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) sufficiently close to the point (x_0, y_0) .

A function of two variables has a absolute maximum (global maximum) at a point (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) on the domain of the function.

Analogously we say that a function $z = f(x, y)$ has a minimum (local) at a point (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) sufficiently close to the point (x_0, y_0) . Similarly we define absolute minimum (global minimum).

The maximum and minimum of a function are called extrema of the function; we say that a function has an extremum of a given point if it has a maximum or minimum at the given points.

Example 10.1. The function $z = (x - 1)^2 + (y - 2)^2 - 1$ contains a minimum at $x = 1, y = 2$.

Solution: As $f(1,2) = -1 < f(x,y)$ for all $x \neq 1$ and $y \neq 2$ i.e.,

$$f(x,y) > f(1,2) = -1$$

Example 10.2 The function $z = \frac{1}{2} - \sin(x^2 + y^2)$

Solution:

For $x = 0, y = 0, f(0,0) = \frac{1}{2}$. Now for $0 < x^2 + y^2 < \frac{\pi}{6}, \sin(x^2 + y^2) > 0$.

So $f(0,0) > f(x,y), 0 < x^2 + y^2 < \frac{\pi}{6}$ i.e., $x = 0, y = 0$ is a maximum point of z .

Necessary Conditions for an Extremum: If a function $z = f(x,y)$ attains an extremum at $x = x_0$ and $y = y_0$, then each first partial derivative $(f_x, f_y)|_{(x_0, y_0)}$

either vanishes for these values or does not exist.

This result is not sufficient for investigating the extreme points, but permits finding these values for cases in which we are sure of the existence of a maximum or minimum. Otherwise more investigation is required.

Example 10.3. Consider the function $z = x^2 - y^2$

Solution:

The function has partial derivatives as $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$ which vanish at

$x = 0$ and $y = 0$. But this function has neither maximum nor minimum at

$x = 0$ and $y = 0$, since it takes both negative and positive values. Points at

which $\frac{\partial z}{\partial x} = 0$ (or does not exist) $\frac{\partial z}{\partial y} = 0$ (or does not exist) are called critical

points of the function $z = f(x, y)$. Thus if a function has an extreme point this

can occur at the critical point. Converse may not true.

For investigation of a function at critical points, let us establish sufficient conditions for the maximum of a function of two variables, which can be generalized to functions of more than two variables also.

Theorem 10.1: Let a function f have continuous second partial derivatives on

an open region containing a point (a, b) for which $f_x|_{(a,b)} = 0$ and $f_y|_{(a,b)} = 0$.

Let

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

or

$$d = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)},$$

[$f_{xy} = f_{yx}$ as f has 2nd order continuous partial derivatives]

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b) .
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b) .
3. If $d < 0$, then f has neither a local minimum nor a local maximum at (a, b) .
4. The test is inconclusive if $d = 0$. (Additional investigation is required)

Proof follows from Taylor's theorem.

Note that if $d > 0$, then $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have same sign. This means that $f_{xx}(a, b)$ can be replaced by $f_{yy}(a, b)$.

Example 10.4 Find the extreme point of

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1$$

Solution:

$$\frac{\partial f}{\partial x} = -3x^2 + 4y = 0, \frac{\partial f}{\partial y} = 4x - 4y = 0,$$

solving we obtain $x = y$. i.e., $3x^2 - 4x = 0$ or $x(3x - 4) = 0$. So $(0,0)$ and

$(\frac{4}{3}, \frac{4}{3})$ are the critical points. $f_{xx} = -6x$, $f_{yy} = -4$, $f_{xy} = 4$.

$$d = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 16 < 0.$$

i.e., f has neither minimum nor maximum at critical point $(0,0)$. Hence $(0,0)$ is

a saddle point. We will consider the critical point $(\frac{4}{3}, \frac{4}{3})$

$$d = f_{xx}(\frac{4}{3}, \frac{4}{3})f_{yy}(\frac{4}{3}, \frac{4}{3}) - [f_{xy}(\frac{4}{3}, \frac{4}{3})]^2$$

$$= -\frac{24}{3}(-4) - 16 = 16 > 0,$$

and $f_{xx}(\frac{4}{3}, \frac{4}{3}) = -8 < 0$, we conclude that $f(x, y)$ has a maximum at $(\frac{4}{3}, \frac{4}{3})$

References

W. Thomas, Finny (1998). Calculus and Analytic Geometry, 6th Edition,
Publishers, Narsa, India.

Jain, R. K. and Iyengar, SRK. (2010). Advanced Engineering Mathematics, 3rd Edition Publishers, Narsa, India.

Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

Piskunov, N. (1996). Differential and Integral Calculus Vol I, & II, Publishers, CBS, India.

Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.