Lesson 1

Rolle's Theorem, Lagrange's Mean Value Theorem , Cauchy's Mean Value Theorem

1.1 Introduction

In this lesson first we will state the Rolle's theorems, mean value theorems and study some of its applications.

Theorem 1. 1 [Rolle's Theorem]: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there exists at least one number c in (a,b) such that $f'(c)=0$.

Proof: Assume $f(a)=f(b)=0$. If $f(a)=f(b)=k$ and $k=0$, then we consider $f(x)-k$ instead of $f(x)$. Since $f(x)$ is continuous on $[a,b]$ it attains its bounds: Let M and m be both maximum and minimum of $f(x)$ on $[a,b]$. If $M=m$, then $f(x)=m$ is throughout i.e., $f(x)$ is constant on $[a,b] \Rightarrow f'(x)=0$ for all x in $[a,b]$. Thus $\exists$ at least one c such that$f'(c)=0$. Suppose $M \neq m$. If $f(x)$ varies on (a,b) then there are points where $f(c)>0$ or points where $f(c)<0$ .Without loss of generality assume $M>0$ and the function takes the maximum value at $x = c$, so that $f(c) = M$. It is to be noted that if $c = a$, $f(c) = f(a) = 0 = f(b)$, which is a contradiction. Now as $f(c)$ is the maximum value of the function, it follows that $f(c + \Delta x) - f(c) \leq 0$, both when $\Delta x > 0$ and $\Delta x < 0$.

Hence,

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0$$

when $\Delta x > 0$

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0$$

when $\Delta x < 0$. Since it is given that the derivative at $x = c$ exists, we get $f'(c) \leq 0$ when $\Delta x > 0$ and $f'(c) \geq 0$ when $\Delta x < 0$. Combining the two inequalities we have, $f'(c) = 0$.

Note: Rolle's theorem shows that b/w any two zero's of a function $f(a)$ there exists at least one zero o $\acute{f}(x)$ i.e., $f(a) = f(b)$ clearly f is continous on [-1,1]

Example 1: Verify the Roll's theorem for $f(x) = x^2 \quad for\ all\ \ x \in [-1,1]$.

Solution:

(i) $f(1) = f(-1) = 1$, (ii) f is differentiable on $[-1,1]$, so all conditions of Roll's theorems are satisfying. Hence $f'(c) = 2c = 0$ implies $c = 0$ and $c \in (-1,1)$.

Example 2: $f(x) = 1 - |x|$ in $[-1,1]$.

Solution:

$f(-1) = f(1) = 0$, f is continuous. But $f(x)$ is not differentiable at $x = 0$.

Note that $f'(x) \neq 0$, for which $f(x)$ is differentiable. As $f'(x) = -1$, for $x > 0$ and $f'(x) = 1$, for $x < 0$.

Example 3: Show that the equation $3x^5 + 15x - 8 = 0$, has only one real root

Solution:

$f(x) = 3x^5 + 15x - 8$ is an odd degree polynomial, hence it has at least one real root as complex roots occurs in pair.

Suppose $\exists$ two real roots x_1, x_2 such that $x_1 < x_2$, then on $[x_1, x_2]$, all properties of Roll's theorem satisfied, hence $\exists\ c \in (x_1, x_2)$, such that $f'(c) = 0$, But $f^{'}(x) = 15x^4 + 15 = 15(x^4 + 1) > 0, \quad for\ all\ x$, a contradiction to Rolle's therorem. Hence the equation has only one real root.

1.2. Mean Value Theorems

Theorem 1.2 [Lagrange's Mean Value Theorem]: If a function $f(x)$ is continuous on $[a, b]$, differentiable (a, b), then there exists at least one point c, $a < c < b$ such that $f(b) - f(a) = f^{'}(c)(b - a)$. Hence Lagrange's mean value theorem can be written as

$f(b) - f(a) = hf^{'}(a + \theta h)$, where $h = b - a\,; 0 \leq \theta \leq 1$.

Geometrical Representation: If all points of the arc AB there is a tangent line, then there is a point C between A and B at which the tangent is parallel to the chord connecting the points A and B.

1.2.1 Cauchy's Mean Value Theorem

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem.

Theorem 1.2 [Cauchy's Mean Value Theorem]: It states that if functions f and g are both continuous on the closed interval $[a, b]$, and differentiable on the open interval (a,b) and $g(a) \neq g(b)$ then there exists some $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note 1: Cauchy's mean value theorem can be used to prove L'Hospital's rule. The mean value theorem (Lagrange) is the special case of Cauchy's mean value theorem when $g(t) = t$.

Note 2: The proof of Cauchy's mean value theorem is based on the same idea as the proof of the mean value theorem

1.2.2 Another form of the statement: If $f(x)$ and $g(x)$ are derivable in $[a, a+h]$ and $g'(x) \neq 0$ for any $x \in [a, a+h]$, then there exists at least one number $\theta \in (0,1)$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad (0 < \theta < 1)$$

Example 4: Write the Cauchy formula for the functions $f(x) = x^2$, $g(x) = x^3$ on $[1,2]$.

Solution:

Clearly $f and\ g\ \ are\ continous\ and\ diff.on\ [1,2]$ $g'(x) = 3x^2 = 0$ iff

$x = 0$, $0 \notin [1,2]$. $f'(x) = 2x$. Hence $g(1) \neq g(2)$

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}.$$

i.e., $\frac{4-1}{8-1} = \frac{2c}{3c^2}$ implies $\frac{3}{7} = \frac{2}{3c}$, so $c = \frac{14}{9}$.

1.2.3 The Intermediate Value Theorem It states the following: If $y = f(x)$ is continuous on $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = N$.

1.2.4 Applications of the Mean Value Theorem to Geometric properties of Functions.

Let f be a function which is continuous on a closed inteval $[a, b]$ and assume f has a derivative at each point of the open interval (a, b). Then we have

1. (i) If $f'(x) > 0 \quad for\ all\ \ x \in (a, b)$, f is strictly increasing on $[a, b]$.
2. (ii) If $f'(x) < 0 \quad for\ all\ \ x \in (a, b)$, f is strictly decreasing on $[a, b]$.
3. (iii) If $f'(x) = 0 \quad for\ all\ \ x \in (a, b)$, f is constant.

Intermediate value Theorem for Derivatives: If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$ then for any number d between $f'(a)$ and $f'(b)$ there is a number $a < c < b$ where $f'(c) = d$.

Application: If $f'(x)$ exists with $f'(x) \neq 0$, on any interval then f has a differentiable inverse, there.

Converse of Rolle's theorem : - (need not true).

Example 1.5 Let $f(x)$ be continuous on $[a, b]$ and differentiable (a, b). If $\exists c \in (a, b)$ such that $f'(c) = 0$, does it follow that $f(a) = f(b)$?

Solution:

No: Take for example $f(x) = x^2$ on $[-1,2]$, $f'(x) = 2x = 0$ implies $x = 0$. But $f(-1) = 1$ and $f(2) = 4$.

Example 1.6 Show that $|\sin x - \sin y| \leq |x - y|$

Solution:

Let $f(t) = \sin t$ on $[y, x]$, By mean value theorem $\sin x - \sin y = f'(c)(x-y)$,

But $f'(t) = \cos t$, and $|\cos t| \leq 1$, for all t. Hence

$|\sin x - \sin y| = |f'(c)(x-y)| \leq |x-y|$.

Example 1.7 Show that $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$, for all $x_2 > x_1$.

Solution:

Let $f(x) = \tan^{-1}x$ on $[x_1, x_2]$. By mean value theorem $\tan^{-1}x_2 - \tan^{-1}x_1 =$

$f'(c)(x_2 - x_1) = \frac{1}{1+c^2}(x_2 - x_1)$ but $\frac{1}{1+c^2} < 1$ for all c. Hence the results.

Questions: Answer the following question.

1. Verify the truth of Rolle's theorem for the functions

   (a) $f(x) = x^2 - 3x + 2$ on $[1,2]$

   (b) $f(x) = (x-1)(x-2)(x-3)$ on $[1,3]$

   (c) $f(x) = \sin x$ on (a) $[0, \pi]$

2. The function $f(x) = 4x^3 + x^2 - 4x - 1$ has roots 1 and -1. Find the root of the derivative $f'(x)$ mentioned in Rolle' s throrem.

3. Verify Lagrange's formula for the function $f(x) = 2x - x^2$ on $[0,1]$.

4. Apply Lagrange theorem and prove the inequalities

   (i) $e^x \geq 1 + x$ (ii) $\ln(1+x) < x \;\; (x > 0)$

(iii) $b^n - a^n < nb^{n-1}(b-a)$ for $(b > a)$

5. Using Cauchy’s mean value theorem show that $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Keywords: Rolle’s Theorem, Lagrange's and Cauchy’s mean value; L'Hospital's rule; Intermediate value.

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Suggested Readings

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